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STRESS ANALYSIS OF THIN AND THICK PLATES ON ELASTIC FOUNDATIONS USING BOUNDARY AND FINITE ELEMENT METHODS

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To

My Mother, Father,

Brothers, Sisters,

my Wife,

and my Childern,

Abdulla and Maytha

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ABSTRACT

In this work an attempt has been made to derive a full finite element and boundary element theory for the analysis of thin and thick plates on elastic foundations. A new high order shear finite element capable of the analysis of thin thick plates has been derived using Hermitian and Lagrangian shape functions. Different new boundary element derivations for the analysis of thin plates on elastic foundations are introduced using 3 degrees-of-freedom per node. A full new derivation of boundary elements for thick plates on elastic foundations using complex Bessel functions is presented. Fourier and Hankel integral transforms have been employed for the derivation of different fundamental solutions required for boundary element analysis. Several techniques for dealing with singular and divergent integrals encountered with boundary integral equations were developed including the use of "Modified Kelvin Functions" and fictitious boundary concept. Some case studies with different loading and boundary conditions were tested and proved that the new derivations presented in this work are correct and reliable for the analysis of thin and thick plates on elastic foundations.

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CHAPTER ONE

INTRODUCTION

1.1 GENERAL INTRODUCTION

Foundations have been the concern of mankind in the erection of its permanent structures. There is no structure which has ever survived the destruction of its foundation unless they are replaced or strengthened, and those who survive through the ages have done so because of their foundations.

The analysis of the interaction between structural elements such as beams, plates or shells and the elastic media is of great importance to several branches of engineering. Thin and thick plates are frequently used as components of aircraft and other structures. One of the functional requirements is to reduce the deflection and stresses of such plates which was usually achieved by providing the plate with a proper elastic support such as an elastic foundation.

Structures such as buildings, bridges and concrete dams are designed to be supported by earth. These structures consist of two parts, the superstructure or the upper part and the substructure element which is the interface of the superstructure and the supporting ground. The foundation can be defined as the substructure and the adjacent zone of soil or rock which will be affected by both substructure element and its load.

The practice of foundation engineering was largely empirical until 1925 when **K. Terzaghi** [Ref 1, 2] often called the **"The father of soil mechanics"** published his book **"Erdbaumechanik Auf Bodenphsikalscher Grundlage"**, and from 1950 onwards foundation engineering has developed into a more rational approach.

Foundations for structures such as buildings from the smallest residential to the tallest high-rise and bridges are for the purposes of transmitting the superstructure loads. These loads come from column type members with stress intensities ranging from 10 mpa for concrete to 140 mpa for steel, to the supporting capacity of the soil which is of the order of 200 kpa.

Foundations could be used for several purposes such as buildings, radio and television towers, bridge peers industrial equipment, port or mats of marine structures, impact machinery turbines and nuclear plant facilities.

A foundation engineer must be versed in both the geotechnical aspect of soils as well as the structure behaviour produced by complex foundation-soil interaction and concerned directly with the structural member which affects the transfer of load from the superstructure to the soil such that the resulting soil stability and estimated deformation are tolerable.

The foundation engineer has a particular interest in the following properties:

1. Strength parameters (stress-strain modulus, shear modulus, Poisson's ratio).
2. Compressibility index (for deformation/settlement).
3. Permeability.
4. Gravimetric-volumetric data (unit weights, specific gravity, void ratio, water content).

The interaction between the foundation and the soil upon which it is resting is very important in the analysis of most foundation engineering problems. A simplified model referred to as the **Winkler model** [Ref 3] has been used to analyse the soil structure interaction.

The wide use of comparatively thin plates, slabs and shells in modern structures has brought about advances in the theory of plates and shells. Many practical engineering problems fall into categories of "plates" and "shells". Engineering components or structures with a thickness very much smaller than their other dimensions are often described as either plates or shells. Conceptually, a plate is a thin flat body, symmetric with respect to a plane for instance, the (x,y) plane, so that particles in the plate are identified by coordinates (x,y,z) with $-h/2 \leq z \leq h/2$, $h=h(x,y)$, being the thickness of the plate. There are a variety of examples of plates such as table tops, street manhole covers, side panels and roofs of buildings, turbine disks and tank bottoms.

The bending properties of a plate depend greatly on its thickness as compared with its other dimensions. Plates may be classified into two groups: Thin plates with small and large deflection and thick plates. The simplest type of plate bending finite element has rectangular geometry and according to the criterion applied to thin plates, the ratio of the thickness to the smaller span length should be less than 1/20.

The development of shells as a structural form has indeed added an exciting chapter to the civil engineering construction, and the past five decades have witnessed several bold and daring instances of the use of thin reinforced concrete shells, in a variety of forms in roofs all over the world. Shells are structures which derive their strength from "form" rather that "mass" which enable them to put a minimum of material to maximum structural advantage. In general a shell structure may be defined as the solid material enclosed between two surfaces , the distance between them being the thickness of the shell. If the thickness is small compared to the overall dimension of the boundary surfaces then the shell is defined as "thin" shell, if not it is termed "thick".

Even though shells have been enjoying a wide use in roofs, they are newcomers to the family of structural foundations and it is less than three decades only since Candela [Ref 4] poured his first shell footings into the Mexican soil. Shells could be used in a variety of things and such examples are pressure vessels, aeroplane wings, pipes, the exterior of rockets, missiles, automobile tyres, incandescent lamps, caps, roof domes, factory and car sheds.

The theory of plates and shells on elastic foundations occupies a prominent place in contemporary structural mechanics. A very large number of studies have been devoted to this subject, and valuable practical methods for the analysis of plates and shells on elastic foundations have been worked out. Approximate methods are obviously best suited for analysing complex three dimensional engineering structures on elastic foundations.

In practice, solutions to any real or complex problem can only be obtained via some form of numerical analysis for which the principal techniques available fall into two main categories: differential methods and integral methods. Matrix methods of structural analysis first proved of interest in the late forties and early fifties by S. Levy [Ref 5,6] and J. Argyris [Ref 7,8] as a result of a requirement for high-strength aircraft structures. The lack or complete absence of computational power of high speed digital computers was a problem until 1947, in fact the first machine did not appear until 1951. Plate bending problems are usually analysed by finite difference methods (F.D.M) or by finite element methods (F.E.M) or more recently, by boundary element methods (B.E.M). The finite element method first started around the mid 1950's which is now considered as the most powerful method for the approximate solution of boundary value problems. The finite element method is a numerical procedure for solving engineering boundary value problems within specified domains with the aid of piecewise discretisation concepts. The domain of the problem is divided into a suitable number of subdomains or elements connected by specified nodes. The equation for each element can be formulated using a suitable energy principle. Then the equations of the whole domain can be assembled and solved. However, there are difficulties with the use of the F.E.M in the analysis of complex structures, due to the generation of thousands of equations which would probably require an enormous amount of computer CPU time for their solution.

The difficulties encountered in the use of finite element method were responsible for the introduction and development of another numerical method which is the boundary integral equation or the so called boundary element method. Although integral equation techniques for continuum mechanics problem have long been studied by mathematicians, it is only comparatively recently that they become attractive for solving practical engineering problems. However, a wide range of engineering problems in structural mechanics and electrical engineering can be formulated with boundary integral equations, these equations are most conveniently solved by numerical methods. The numerical techniques for integral equations have been developed rapidly and many articles were written on the subject in the past decade. An integral equation within a domain can be transformed into what is known as a boundary integral equation (B.I.E) over the boundary of the domain. A B.I.E may be solved by means of piecewise discretisation where the boundary is divided into sub-boundaries or (boundary elements) then the equation of the boundary pieces are assembled to form a system of algebraic equations. The solution to such equations involves the evaluation of some boundary integrals and this approach is called the boundary element method (B.E.M). The B.E.M has many advantages compared with other numerical approaches as follows:

(i) It reduces the dimension of the problem which results in smaller system of equations and a considerable reduction in the data required.

(ii) The B.E.M offers continuous interior modelling within the solution domain.

(iii) The method is well suited to problems of infinite domain such as soil mechanics, hydraulics, stress analysis.

1.2 OBJECTIVES OF THE RESEARCH

The objectives of this work is to develop a boundary element and finite element package capable of solving engineering problems for thin and thick plates on elastic foundations. In this study a comparison between the finite element and boundary element method will be carried out using several case studies with known analytical solutions.

The theory and fundamental solution of the boundary integral equations for thin and thick plates on elastic foundation will be derived. Accuracy measures will be developed so as to make the boundary element solution valid for plates with irregular shapes.

1.3 THESIS LAYOUT

After this introductory chapter, a comprehensive literature review is summarised in chapter (2) which covers finite and boundary element analysis of plates and shells on elastic foundations.

The theory of finite element analysis of thin and thick plates on elastic foundations is reviewed in chapter (3) where Kirchhoff, Mindlin and a new high-order plate bending elements are described.

A new derivation for boundary element analysis of thin plates on elastic foundations based upon three degrees-of-freedom per node is introduced in chapter (4) which contains also a new idea for overcoming singular integration by means of "modified" Kelvin functions introduced in this work. Two new different ideas for the reduction of domain-loading integral terms into simple boundary integral terms are also discussed.

The full derivation of boundary element analysis of thick plates on elastic foundations is introduced for the first time in chapter (5) which explains the use of a fictitious boundary outside the domain to overcome singular and divergent integrals encountered in the boundary integral equations. The reduction of domain-loading terms to boundary integral terms is also described.

A short review of the finite element and boundary element programming package developed in this work is given in chapter (6) which indicates that 3 finite element programs and 6 boundary element programs have been coded.

In chapter (7), case studies of thin and thick circular and rectangular plates under uniformly-distributed and concentrated loading have been tested using developed finite and boundary element programs and the results obtained were assessed against analytical solutions from published work and from solutions developed in this work.

The final conclusion and recommendations for future work are summarised in chapter (8).

CHAPTER TWO

LITERATURE REVIEW

2.1 FINITE ELEMENT METHOD

In modern times the idea of piecewise discretisation has been founded in aircraft structure analysis where, for example, wings and fuselages are treated as assemblages of strings, skins and shear panels. However, beginning in 1906 and sporadically there after, researchers suggested the "**Lattice Analogy**" to solve continuum problems [Ref 8,9], where the continuum is approximated by a regular mesh of elastic bars. In 1941 **Hernikoff** [Ref 10] introduced the so-called framework method, in which a plane elastic medium was represented as a collection of bars and beams. The use of piecewise continuous functions defined over a subdomain to approximate the unknown function dates back to the work of **Courant** [Ref 11] who used an assemblage of triangular elements and the principle of minimum potential energy to study the St-Venant torsion problem.

By 1953, engineers were writing stiffness equations in matrix notation and solving the equations with digital computers [Ref 12]. In 1956, a major break-through has been made in numerical methods for structural mechanics when **Turner et al** [Ref 13] published their well-known work in which complex in-plane plate problems were solved by using triangular elements each of which was described by three corner nodes. **Clough** (1960) was the first to use the term "**Finite Element**" in his paper " **The finite element method in plane stress analysis**" [Ref 14].

By 1963 the method was recognised as rigorously sound, and it became a respectable area for research [Ref 15]. In 1965, **Melosh** [Ref 16] realised that the finite element method could be expanded to field problems using variational methods and due to the importance of this contribution **Zienkiewicz** and his co-workers applied it to a large number of steady-state and transient field problems [Ref 17]. Many papers on the application of finite element method to problems of heat conduction and seepage flow appeared in 1965.

Since then large general-purpose finite element computer problems emerged during the late 1960's and early 1970's such examples include **ANSYS**, **NASTRAN**, and **PAFEC** where each of these programs contains several kinds of elements and performs static, dynamic and heat transfer analysis. Additional capabilities such as preprocessing and postprocessing have been added with the aid of graphics which makes it easier, faster, and cheaper to carry out a finite element analysis. In the early 1980's graphics development became intensive as hardware and software for interactive graphics became available and affordable.

Engineers and mathematicians worked with finite elements in apparent ignorance of one another [Ref 18]. Since then, the literature on "**Finite-element**" applications has grown exponentially and today there are numerous journals which are devoted to the theory and application of the finite element method including some useful description of applications of the finite element method. A detailed review is given by **Zienkiewicz** [Ref 19,20,21] for a brief history of the finite element method including some useful description of applications in the field of engineering.

2.2 BOUNDARY ELEMENT METHOD

The boundary element method (B.E.M) has been progressively used for the solution of engineering problems in the last few years. The recent development of the boundary element method attracts wide attention from viewpoints of less computer cost for numerical calculation and of high accuracy of its solution. The B.E.M, or boundary integral equation method (B.E.I.M) as named before, originates in the theory of integral equations and based on the integral representations for the solution of the partial differential equations where the first results dated back to (1903) when **Fredholm** as quoted in [Ref 22], published his rigorous work on integral equations encountered in potential theory. The classical work of **Kellogg** [Ref 23], in (1929) gave a complete review of the applicability of integral equations to the field theory. The notable works of the Russian author **Mikhlin** [Ref 24,25] appeared in English in 1957 and 1965 where he mainly concentrated on the singularities and discontinuities of integral equations.

The broad use of the computer in the early 1960's gave rise to the B.E.M with **Symm**, **Jaswon** and **Cruse** ranking among its pioneers [Ref 26]. Since the 1960's researchers have managed to apply the boundary element method to a large range of engineering applications.

The boundary element methods (B.E.M) can be classified into three main groups: Indirect, Semi-direct and Direct. In the **indirect** method, the discretised integral equations are formulated in terms of "**Fictitious**" distributions of the singular solutions over the problem boundaries, and a review of the early works can be found in [Ref 27]. In the **semi-direct** method the integral equations are formulated in terms of unknown functions which are related to stress functions, [Ref 27]. The **direct** method, where the unknown functions in the integral equations are in fact the physical tractions and displacements on the boundaries, is mainly used in this work. The solution of the integral equations yields all the stresses and displacements on the system boundaries directly obtained by numerical integration. Algorithms based upon this formulation have been described by many authors, see [Ref 27].

During the 1970's, developments made in the analysis of the finite element method have started to find their way into the formulation and solution of boundary integral equations and the first book titled "**Boundary Elements**" was published by **Brebbia** in 1978. In 1981, the first comprehensive work on the B.E.M and its applications in the various field of engineering science have been presented by **Banerjee** and **Butterfield** [Ref 27]. The boundary element literature is increasingly becoming fertile in textbooks devoted to specialised engineering subjects, such as those of **Mukherjee** [Ref 28] on creep and fracture, **Crouch** and **Starfield** [Ref 29] on solid mechanics, **Liggett** and **Liu** [Ref 30] on porous media flow and **Verturini** [Ref 31] on geomechanics. Since that day onward several books explaining the fundamental concepts of boundary elements method [Ref 32-38], and international conferences on the topic of boundary element method were held in 1978 to date [Ref 39-47]. Another series of texts have also been published by **Banarjee et al** since 1979 [Ref 48-51].

2.3 PLATES AND SHELLS

The stress analysis of plate bending problems began in the 19th century when **Sophie Germain** presented her classic work with **Lagrange** [Ref 52]. **Sophie Germain** was the first to use mathematical theory in 1811 and she showed that for the case of rectangular plate, her analytical results correlated well with experimental results. In the following forty years, many new points were made by great researchers such as **Navier, Poisson, Cauchy and Kirchhoff. Poisson** was the first to consider the case of a circular disc with a radially symmetric load.

In 1850 **Kirchhoff** showed that only two boundary conditions were required and his results were in agreement with **Poisson's**. There are many short comings in **Kirchhoff's** plate bending theory and up till now people still researching for more general plate bending formulations. In particular, **Kirchhoff's** theory ignores the effect of thickness by neglecting the transverse shear stresses throughout the thickness of the plate. **Reissner** [Ref 53] and **Mindlin** [Ref 54] had carried out interesting work in this area.

Many approximate methods have been attempted and they must be evaluated in terms of their accuracy, efficiency and their flexibility for the use in different plate problems. The first numerical results were obtained by **B.M. Koyalovich** in 1902 and between then and 1947 three main techniques had been evolved due to **Hencky** [Ref 55,56,57], **Ritz** [Ref 58], and **Marcus** [Ref 59]. **Hencky's** technique is a series method; the coefficients are obtained by partially inverting an infinite matrix. The **Ritz** method is an energy method and it is essentially used in finite element method. The **Marcus** method is probably the most flexible which uses the separation of variables technique with approximations.

Plate bending was one of the problems to be solved by analytical methods as explained by **Love** [Ref 60] and **Timoshenko** [Ref 61] who published a short review on the use of finite difference method which was the first numerical method to be applied to plate bending problems. However, with the appearance of the finite element method in the mid 1950's researchers started applying it to the solution of plate bending problems. The finite elements for plate-bending are classified into two groups **conforming** and **non-conforming** elements, **Melosh** [Ref 16], **Zienkiewicz** and **Morgan** [Ref 21] derived one of the early non-conforming elements. A generalised approach for the derivation of **Hermitian Shape Functions** for triangular and quadrilateral elements has recently been published by **El-Zafrany and Cookson** 1986 [Ref 62,63]. Many interesting finite element books which contain plate bending analysis have been published, such as those by, **Zienkiewicz** [Ref 17], **Cook** [Ref 64], **Hinton** and **Owen** [Ref 65,66], **Irons** and **Ahmad** [Ref 67], **Wait** and **Mitchell** [Ref 8], and many others.

An enormous effort has been devoted to the development of the finite element method for the analysis of plate bending. The literature in plate bending using finite elements is extensive but the majority of the recent work appears to be involved with the development of the Mindlin plate elements. Mindlin's theory is preferred to that of Kirchhoff's thin theory for many reasons. One important advantage is that many Mindlin plate elements require only $C^{(0)}$ continuity of the lateral displacement "w" and the independent slopes θ_x and θ_y (rotations about the X-axis and Y-axis) where elements that are based on the classical Kirchhoff thin plate theory require $C^{(1)}$ continuity. If transverse shear effects are present in the problem, they are automatically modelled with Mindlin elements. A disadvantage of Mindlin elements, when used for thin plates, is the phenomenon known as shear locking. **Lee et al** [Ref 68], stated that transverse shear effects can cause over-constraining or locking which limits the ability of the finite element model to represent properly the bending behaviour of a thin plate. The usual and most popular method of avoiding shear locking is to use selective or reduced integration schemes. **Lee et al** [Ref 68], also used different approaches to this problem where they applied a mixed formulation based on a modified **Hellinger-Reissner** principle and the **Reissner-Mindlin** theory for plate bending.

Prathap [Ref 69] introduced a piece of research which shows that a right-angled triangle correctly aligned and with correctly chosen integration points will not lock. **Fricker** [Ref 70] introduced a new improved three noded triangular element for plate bending. Another approach to the development of the plate bending elements for thin plates involves the so-called "**Discrete Kirchhoff Hypothesis**". In this approach the classical equations are abandoned in favour of a bending theory which includes shear deformations. The result is that only $C^{(0)}$ continuity is required for the shape functions and to capture the behaviour of thin plate theory, the constraint of zero shear strains is imposed at discrete number of points. **Dhatt et al** [Ref 71] presented an extensive paper describing a new triangular plate/shell element which has been developed using the method described above.

An extensive work on quadrilateral elements has been developed and it is detailed in many papers. **Hughes et al** [Ref 72] have presented a four noded quadrilateral element with three degrees-of-freedom per node. **Prathap** [Ref 73] offered a similar derivation for a quadrilateral thin plate bending element except that an optimal set of **Kirchhoff** constraints are enforced at the centre of the element by a penalty function type formulation.

A proposed quadratic **Mindlin** element which uses a rarely seen uniform two-point Gaussian integration system is given by **Crisfield** [Ref 74]. When a reduced order of numerical integration of the stiffness terms is used on some isoparameteric displacement based **Mindlin** plate elements, improved behaviour is often obtained.

A reduced integration rule used for the evaluation of the stiffness matrix associated with the troublesome shear energy is given in [Ref 75,79]. **Tsach** [Ref 76] introduced an analysis of a sufficient and necessary criteria of selective reduced integration. **Carpenter et al** [Ref 77] gave a description of locking and shear scaling factors in $C^{(0)}$ bending elements with reduced integration. **Pugh et al** [Ref 78], studied the behaviour of quadrilateral plate bending elements with reduced integration and introduced the concept of a "penalty function" in the shear components of the variational statement to force the elements in the thin range to adhere to the constraints of Kirchhoff's theory thus preventing any locking from occurring.

Hughes et al [Ref 75] developed plate elements with reduced and selective integration and claimed that the element developed in [Ref 74] behaved poorly in the thin plate limit.

Using Mindlin theory, other higher order elements have been derived such as **Reddy elements** [Ref 80], which have been derived for non-linear applications and the **El-Zafrany/Cookson element** [Ref 81], which has been derived using Mindlin theory and Hermitian interpolation. Both elements work well over a large thickness range. Along the years, plate bending problems have been analysed using finite difference or finite element method but nowadays it can be solved using the boundary element method and investigations in recent years have been carried out to solve such problems. **Jaswon and Maiti** [Ref 82] produced the first plate bending application of boundary integral treatment for uniformly loaded clamped and simply supported plates. **Altiero and Sikarskie** [Ref 83] have presented a technique which consists of embedding a real plate within a fictitious one using the indirect method and due to the complexities associated with the boundary conditions only the clamped plates have been investigated. **Tottenham** [Ref 84] presented an integral formulation for shallow shells using a similar approach. The direct formulation for plate bending have been introduced by many authors such as **Bezine, Stern, Costa and Brebbia. Bezine** [Ref 85,86] presented an integral formulation using a constant element, but this formulation results in divergence of the integral equations on the boundary and this is due to the fact that the divergent integral which appears in plate bending formulation was not taken into account in the derivation of integral equations for a point on the boundary. **Stern** [Ref 87,88] suggested a general formulation for solving plate bending problems, but his formulation does not take into account the possibility of discontinuities of the boundary conditions.

Recently, a considerable amount of research on plate-bending problems has been carried out at Cranfield. A three degrees-of-freedom formulation for thin plate bending problems was introduced by **M. Debbih** [Ref 89], and the work was extended to thick plates with interesting solutions for singular-integration problems [Ref 90].

In the early 1960's, a shift was seen in application from aircraft to space vehicles and this change resulted in the use of new highly-curved unstiffened thin shell structures. The finite element method in 'thin-shell structural analysis' began with applications to aerospace vehicles and the literature contains numerous descriptions of these applications as indicated in a collection of papers given by [Ref 91]. By 1969, a large number of curved thin shell finite elements had appeared in the literature. During the past fifteen years or so there has been a considerable activity in the design of nuclear reactor structures using solid or thick-walled shell structures. It is clear from the literature that there are three distinct approaches to the finite element analysis of thin-shell structures, which are ; the facetted form using flat elements as an approximation of the geometry, curved shell elements formulated directly from appropriate thin shell theories and isoparametric solid elements specialised to tackle shells by applying, in discrete form appropriate thin shell assumptions (for example Kirchhoff normality hypothesis). The majority of references on finite element methods applied to shell problems fall into the latter two categories.

Ahmad et al [Ref 92] presented an extensive paper discussing the analysis of thick and thin shell structures by curved finite elements which overcomes the approximations to the geometry of a structure and shear deformation.

A paper discussing some modifications of an isoparametric shell element has been given by **Takemoto et al** [Ref 93], where they showed the effects of adding internal freedoms to the element. **Parish** [Ref 94] gave a survey of the 9-noded degenerated, reduced integration, shell element.

Hansen and Heppler [Ref 95] proposed a Mindlin shell element that satisfies rigid-body requirements. In this work, strain-displacement relations of the Mindlin type derived in shell coordinates for a general shell of arbitrary thickness and geometry are employed in the development of a finite element that can reproduce all rigid-body motion exactly.

Shell elements based upon Mindlin's theory are more accurate for problems in the thick range than those based upon Kirchhoff's theory. Many elements based on Mindlin's thick plate theory only require Lagrangian shape functions which are easier to program where Kirchhoff's theory requires Hermitian shape functions when the thickness is reduced to thin range. However, Mindlin facet shell elements become inaccurate due to shear locking.

The use of facet-shell elements appeared to be quite controversial. **Zienkiewicz and Irons** have different views on the subject, **Bruce Irons** stated in his book [Ref 96] that the ideas of using shells as assemblages of flat plates only lead directly to disaster especially if one works in the field where very thin shells are common. **Zienkiewicz** devoted a whole chapter of his book [Ref 20] to "Shell as an assembly of flat elements". **Bruce Irons** preferred his own seimiLoof shell element which can be found in [Ref 21, 97].

Facet shell elements remain a problem since elements based on either Kirchhoff's or Mindlin's theories are not capable of tackling the full range of thicknesses. Researchers worked on methods of modifying these theories to allow them to analyse thin and thick structures. **Zienkiewicz et al** [Ref 139] developed a method for tackling thin shell structures by using thick shell elements with "reduced integration". This showed that reducing the order of the numerical integration for all stresses gave accurate results in the thin range. No mathematical arguments were presented for explaining why such a phenomenon is happening, only general observations on element behaviour were noted. **Ashwell and Saber** [Ref 91] described a new cylindrical shell finite element which is based on simple independent strain functions.

Meek and Tan [Ref 15] described a discrete Kirchhoff plate bending element with Loof nodes and when superimposed with the linear strain triangle results in a facetted shell element free from deficiencies of singularity.

2.4 FOUNDATION

Soil-structure interaction between the soil foundation and the supported structure is widely recognised as a very important phenomenon in foundation engineering. This interaction is very important for highway transportation problems, where the main interests is in the vertical reaction of the soil foundation on the supported structure. A considerable amount of work has been carried out on the problem of soil structure interaction using the finite element method. The effect of soil-structure interaction is classified into two categories; the direct method and the substructure method. A two dimensional model of a gravity dam on a layered foundation has been analysed by **Wilson** [Ref 98] using the direct method, where the entire system was modelled by means of finite elements and the analysis was carried out in the time domain by a numerical integration of the equations of motion. A similar procedure was employed by **Finn** [Ref 99] using a superposition technique. **Liang** [Ref 100] has studied embedded foundation as well as the interaction between adjacent footings. In the substructure method, soil and building are treated as two different substructures. The soil can then be analysed by any method such as the finite element method or the boundary element method. **Sandi** [Ref 101], **Chopra and Perumalswami** [Ref 102] presented a substructure techniques for the problem of a dam modelled by finite elements while the soil was treated as an elastic half space.

The most widely used thick plate theories are those of **Reissner and Mindlin. Babu, Reddy and Sodhi** [Ref 103] developed a high precision thick triangular orthotropic plate-bending element on an elastic foundation. The element is three noded with twelve degrees-of-freedom per node and takes into account shear deformation and rotary inertia. Thick rectangular plate on Winkler elastic foundation have been investigated by **Naghdi, Rowley and Frederick** [Ref 104,105] using Levy and Navier type solutions.

Dumir and Bhaskar [Ref 106] studied the effect of various parameters on the linear and geometrically nonlinear response of orthotropic immorably simply supported and clamped rectangular plates resting on Pasternak foundations for the case of uniformly distributed loads.

Voyiadjis and Kattan [Ref 107] presented a refined theory for moderately thick plates on elastic foundations. Comparisons are made with the classical plate theory and the corresponding Reissner plate theory as given by **Frederick** [Ref 105].

Currently, the boundary element method is widely used to solve a large range of engineering problems. Its main attraction is that it reduces the dimensionality of the problem. The boundary element method provides an effective approximate numerical technique for the solution of a wide range of plates on elastic foundations. The use of infinite elements to represent the far field in dynamics soil-structure interaction problem was investigated by **Chow and Smith** [Ref 108]. **Wass** [Ref 109] proposed a consistent boundary element applied strictly to layers of soil where an infinitely stiff stratum occurs. **Smith** [Ref 110] proposed a superposition boundary and **Cundall** [Ref 111] proposed refinement of the superposition boundary.

The dynamic behaviour of foundation has been subject to a great interest in recent years. A comprehensive account of the literature on the dynamic response of three dimensional rigid foundation laying on the surface of the soil medium which is modelled as an elastic half-space and subjected to harmonic external forces or seismic waves can be found in **Karabalis and Beskos** [Ref 112]. Analytical work on the dynamic behaviour of flexible foundation has been done by **Oien** [Ref 113] for a strip footing subjected to plane harmonic waves, and by **Schmid** [Ref 114] for rigid circular plates on an elastic half-space subjected to externally applied vertical loads.

The flexural behaviour of plates resting on elastic media is of interst for the design of many engineering problems. **Costa and Brebbia** [Ref 115] applied a technique to solve plates with arbitrary boundary conditions resting on a Winkler-type elastic foundation for which the reaction force is proportional to lateral deflection. The method is formulated by the direct approach and all domain integrals for a uniformly distributed load are transferred to the boundary.

A direct boundary element approach based on Green's identity is employed by **Gospodinov and Ljutskanor** [Ref 116] for static analysis of thin elastic plates. A simple discretisation scheme is described and some numerical examples for static analysis of rectangular plates are given. A computer program for solving a rectangular plate with arbitrary boundary conditions and arbitrary external load has been developed and some results are presented.

Katsikadelis and Armenakas [Ref 117] investigated the B.I.E method with numerical evaluation of the boundary integral equations applied to the bending of simply-supported, thin elastic plate resting on a Winkler-type elastic foundation. The numerical results obtained for circular and rectangular plates are compared with those obtained from **Timoshenko and Woinowsky-Krieger** [Ref 61] analytical solution.

Balas, Sladek and Sladek [Ref 118] presented a boundary integral equation method for solving thin plates resting on a two-parameter foundation. The boundary integral equation solution of a clamped circular plate under a concentrated load is compared to the analytical one.

Two coupled integral equation applied to plates resting on a Winkler-type elastic foundation with arbitrary shape and boundary conditions have been analysed by **Costa and Brebbia** [Ref 115]. The numerical methods presented are compared with their respective analytical solutions, where the approximate results for the various plate bending show an excellent agreement with the exact analytical solution. Circular and rectangular plates were solved for simply supported and clamped boundary conditions to verify the implementation of the above formulation and demonstrate the accuracy of the solution.

Kamiya and Sawaki [Ref 119] presented an approximate boundary element method of the finite deflection analysis of thin elastic plate resting on a Winkler foundation. When the plate deflects the foundation acts as though the elastic spring yielding reaction force were proportional to the deflection perpendicular to the plate. The boundary integral equation was formulated using the generalised Green theorem for a two-dimensional biharmonic differential operator. Numerical solutions for circular and square plates subjected to a uniform load "w" were obtained.

Costa and Brebbia [Ref 120] proposed a way of reducing the domain integrals present in plate bending formulation into boundary integrals for the boundary element formulation of plates on elastic foundation. Numerical results for various plate configurations have been presented using both the boundary technique and domain integral scheme. A circular plate with clamped edges subjected to a uniformly distributed load on an elastic foundation has been solved to validate the formulation and demonstrate the accuracy of using equivalent boundary integrals. The computed results for B.E.M were compared against an analytical solutions given by **Ng** [Ref 121].

Katsikadelis and Kallivokas [Ref 122] has developed a boundary element solution for the analysis of thin elastic clamped plates of any shape resting on a Pasternak-type elastic foundation. A plate with a hole subjected to concentrated loads, line loads and distributed loads have been analysed for deflections, stress resultants, subgrade reaction and reaction on the boundary. Several numerical examples such as circular clamped plate with 32-boundary elements on elastic foundation have been worked out and the results are compared with those of **Timoshenko and Woinowsky-Krieger** [Ref 61] analytical solution. The obtained values differ considerably from the corresponding values for a Winkler-type foundation.

Puttonen and Varpasuo [Ref 123] derived equations based on the direct and indirect boundary element for a plate situated on a one or two parametric elastic foundation. The fundamental solution of the problem is presented as a Fourier-Bessel integral. The boundary element method was compared with the finite element method and it was proven that the boundary element method is well suited for the analysis of a plate on an elastic foundation. A circular plate was analysed for various load conditions using both the Winkler and Pasternak foundation and the results were compared against the analytical solution given by **Timoshenko and Goodier** [Ref 124].

Alessandri and Brebbia [Ref 125] applied the boundary element method to the nonlinear analysis of two masonry walls which were brought to the point of collapse by applying horizontal load simulating the seismic actions. The nonlinear behaviour was modelled by successively correcting the loading and leaving the elastic stiffness matrix unchanged.

Huh and Schmid [Ref 126] considered the boundary integral method to test the dynamic behaviour of a massless foundation on homogeneous soil. The results show that the boundary element methods gives, for surface foundation, good agreement with half-space method and the boundary element method can be applied to more complex problems such as foundation on layered soil as well as to embedded or buried foundations.

An automated approach for the analysis of plates on an elastic foundation is presented by **Dick and Yehia** [Ref 127]. The approach combines automatic mesh generation with non-linear analysis to analyse plates of arbitrary shape on an elastic foundation subject to uplift.

Plate-bending on elastic foundation based on Kirchhoff theory is solved by **Bezine** [Ref 128] using an original boundary integral equation method involving the fundamental solution for plate flexure problems. The fundamental solution is the plate flexure problems and the reactive force of elastic media are compared as load per unit area. Various examples are presented for different boundary conditions, such as cantilever plate, simply-supported plate and clamped plate. The results are compared to the analytical solution and those of **Katsikadelis and Armenakas** [Ref 117].

Kobayashi and Sonoda [Ref 129] analysed a rectangular plate on an elastic foundation using Mindlin's theory. The main attention was paid to twisting moments and shear force distributions along the edges and centre lines of the plates are illustrated graphically to demonstrate the difference between Mindlin's plate theory and classical thin plate theory. The results obtained are compared with the analytical solution found in **Timoshenko and Woinowski-Krieger** [Ref 61].

Paris and DeLeon [Ref 130] has developed an alternative formulation based on the decomposition of the field differential equation into two equations in partial derivatives of second order, and this formulation is dealing with thin plates resting on a Winkler-type elastic foundation under transversal load with different boundary conditions. Two examples are analysed which are a square simply-supported plate along the boundary and resting on an elastic foundation under a uniformly distributed load and a circular clamped plate resting on an elastic foundation under uniformly distributed load. The results obtained are compared with the analytical solution found in **Timoshenko and Woinowski-Krieger** [Ref 61].

2.5 CONCLUSIONS

It is clear from the literature that there are plenty of work dealing with plates and shells and plates on elastic foundations using the finite element method. Published work about thin plates on elastic foundations using the boundary element method contains many problems related to singular integrals and corners. The uses of two degrees-of-freedom per node may lead to inadequate equations for plates with free edge conditions as noticed by Debbih [Ref 90] for ordinary plates. The boundary integral representation of domain loading terms contain divergent integrals which require some modification to the fundamental solution. Unfortunately, to the best of our knowledge, no work has been reported so far on B.E.M solution of thick plates on elastic foundations.

CHAPTER THREE

FINITE ELEMENT ANALYSIS OF PLATES AND SHELLS ON ELASTIC FOUNDATION

3.1 INTRODUCTION

In structure analysis, the term plate is applied to planar structural elements which carry loads applied normal to their plane. Plates are flat, plane surface structures whose thickness is small compared to their other dimensions. The plates can be subdivided into the following categories based on their structural action.

1) Stiff plates, which are thin plates with flexural rigidity, carrying loads mostly by internal moments [Fig. 3.1a].
2) Membranes, which are thin plates without flexural rigidity, carrying the lateral load by central shear forces [Fig. 3.1b].
3) Flexible plates, which represent a combination of stiff plates and membranes and carry external loads by the combined action of internal moments, transverse and central shear forces and axial forces [Fig. 3.1c].
4) Thick plates, whose internal stress condition resembles that of three dimensional continua [Fig. 3.1d].

A facet shell element is a combination of a plate–bending element and an in–plane two–dimensional element based on plane stress theory. There are two basic theories for the analysis of plate bending problems.

a) Kirchhoff's theory [Ref. 61], in which the effects of transverse shears are negligible and is applicable to thin plates, and
b) Mindlin's theory [Ref. 54], which considers approximately the transverse shear effects and is applicable to thick plates.

In this chapter derivations for three different types of elements together with the effect of elastic foundation are provided, which can be summarised as follows:

a) A thin plate element based upon Kirchhoff's theory.
b) A thick plate element based on a first–order shear approximation, similar to Mindlin's element given in [Ref. 54].
c) A new thin–thick plate element based upon a high–order shear approximation, similar to that given by Reissner [Ref. 53].

Using proper rotation matrices, all elements given can be used as faceted−shell elements for the analysis of folded plates and shallow shells.

It is clear from the literature that Kirchhoff's thin plate elements are very accurate for the analysis of thin plates and shells but they are based upon Hermitian shape functions which are difficult to derive. Mindlin's first order plate element provides reasonable accuracy when used for the analysis of thick plates and it is based on Lagrangian shape functions. However, if such elements are used for thin plates, the accuracy may drop drastically due to shear locking [Ref. 76], unless a reduced or selective integration scheme is employed [Ref. 75,131,132]. Hence, the new thin−thick element may provide the best choice since it should have a stable accuracy within a wide range of thickness.

3.2 KIRCHHOFF'S THIN-PLATE ELEMENT

The basic assumptions used for the derivation of Kirchhoff's plate bending element can be summarised as follows:

a) Element material is homogeneous, isotropic and linearly elastic.
b) Displacements are small compared with plate thickness.
c) The normal stresses to the mid-surface of the plate and the transverse shear stresses are negligible.
d) The mid-surface of the plate is a neutral surface during bending.
e) Normals to the mid-surface before deformation remain straight and normal after deformation.

3.2.1 Displacement components

Consider an n−noded element with its mid-plane being in the x−y plane, as shown in [Fig 3.2]. The Cauchy's small−deflection transverse shears at any point(x,y,z) inside the plate can be expressed as follows:

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

From the previous assumptions, the following approximations can be made:

$$w(x,y,z) \simeq w(x,y) \tag{3.1}$$

and γ_{xz}, γ_{yz} are negligible.

Hence, it can be shown that:

$$\frac{\partial u}{\partial z} = - \frac{\partial w(x,y)}{\partial x}$$

and

$$\frac{\partial v}{\partial z} = - \frac{\partial w(x,y)}{\partial y}$$

which can be integrated with respect to" z" as follows:

$$u(x,y,z) = u_o(x,y) - z \frac{\partial w(x,y)}{\partial x} \tag{3.2a}$$

$$v(x,y,z) = v_o(x,y) - z \frac{\partial w(x,y)}{\partial y} \tag{3.2b}$$

where u_o, v_o are the displacement components in x, and y directions, at the neutral bending plane i.e at z=0. For continuity of displacement components u, v, w, the nodal values of u_o, v_o, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ should be specified and used for interpolating u, v, w.

The nodal parameters can, therefore be defined for the n−noded element as follows:

$$\delta_{5n \times 1} = \begin{bmatrix} \underline{\delta}_o \\ \cdots \\ \underline{\delta}_b \end{bmatrix}$$

where

$$\underline{\delta}_{o_{2n \times 1}} = \begin{bmatrix} (u_o)_1 \\ (v_o)_1 \\ (u_o)_2 \\ (v_o)_2 \\ \vdots \\ (u_o)_n \\ (v_o)_n \end{bmatrix}$$

and

$$\underline{\delta}_{b_{3n\times 1}} = \begin{bmatrix} w_1 \\ \left(\frac{\partial w}{\partial x}\right)_1 \\ \left(\frac{\partial w}{\partial y}\right)_1 \\ \vdots \\ w_n \\ \left(\frac{\partial w}{\partial x}\right)_n \\ \left(\frac{\partial w}{\partial y}\right)_n \end{bmatrix}$$

Using Lagrangian interpolation, the components $u_o(x,y)$ and $v_o(x,y)$ can be expressed in terms of their nodal values and Lagrangian shape functions as follows:

$$u_o(x,y) = \sum_{i=1}^{n} (u_o)_i \; N_i(x,y) \tag{3.3a}$$

$$v_o(x,y) = \sum_{i=1}^{n} (v_o)_i \; N_i(x,y) \tag{3.3b}$$

However, the component $w(x,y)$ should be described in terms of the nodal values of "w", $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$, using Hermitian shape functions, i.e

$$w(x,y) = \sum_{i=1}^{n} (f_i \; w_i + g_i \; w_{i,x} + h_i \; w_{i,y})$$

where

$$w_{i,x} = \frac{\partial w}{\partial x} \qquad \text{at node } i$$

$$w_{i,y} = \frac{\partial w}{\partial y} \qquad \text{at node } i$$

and f_i, g_i, h_i are two-dimensional Hermitian shape functions as given in [Ref. 131,132].

Hence, the total displacement components at any point(x,y,z) inside the plate can be expressed as follows:

$$u(x,y,z) = \sum_{i=1}^{n} \left\{ N_i(x,y)(u_o)_i - z\left[\frac{\partial f_i}{\partial x} w_i + \frac{\partial g_i}{\partial x} w_{i,x} + \frac{\partial h_i}{\partial x} w_{i,y} \right]\right\} \tag{3.4}$$

$$v(x,y,z) = \sum_{i=1}^{n} \left\{ N_i(x,y)(v_o)_i - z\left[\frac{\partial f_i}{\partial y} w_i + \frac{\partial g_i}{\partial y} w_{i,x} + \frac{\partial h_i}{\partial y} w_{i,y} \right]\right\} \tag{3.5}$$

$$w(x,y,z) = \sum_{i=1}^{n} \left[f_i\, w_i + g_i\, w_{i,x} + h_i\, w_{i,y} \right] \tag{3.6}$$

3.2.2 Strain–Displacement Relationships

Using Cauchy's strain definition, the strain components can be expressed in terms of displacement components as follows:

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2} \tag{3.7a}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} - z\,\frac{\partial^2 w}{\partial y^2} \tag{3.7b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \right) - 2\,z\,\frac{\partial^2 w}{\partial x \partial y} \tag{3.7c}$$

$$\epsilon_z = \frac{\partial w}{\partial z} = 0 \tag{3.7d}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\,\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0 \tag{3.7e}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\,\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} = 0 \tag{3.7f}$$

Hence, a strain vector can be defined in terms of the basic components of strain as follows:

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \underline{\epsilon}_o + \underline{\epsilon}_b \tag{3.8}$$

where

$$\underline{\epsilon}_o = \begin{bmatrix} \dfrac{\partial u_o}{\partial x} \\ \dfrac{\partial v_o}{\partial y} \\ \dfrac{\partial u_o}{\partial y} + \dfrac{\partial v_o}{\partial x} \end{bmatrix}$$

and

$$\underline{\epsilon}_b = -z \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} = -z\,\underline{\hat{\epsilon}}_b$$

Using equations (3.5) and (3.6), it can be proved that:

$$\underline{\epsilon}_o = \underline{B}_o\,\underline{\delta}_o \tag{3.9}$$

and

$$\underline{\hat{\epsilon}}_b = \underline{\hat{B}}_b\,\underline{\delta}_b \tag{3.10}$$

where

$$\underline{B}_o = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \cdots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \cdots & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

and

$$\underline{\hat{B}}_b = \begin{bmatrix} \cdots & \frac{\partial^2 f_i}{\partial x^2} & \frac{\partial^2 g_i}{\partial x^2} & \frac{\partial^2 h_i}{\partial x^2} & \cdots \\ \cdots & \frac{\partial^2 f_i}{\partial y^2} & \frac{\partial^2 g_i}{\partial y^2} & \frac{\partial^2 h_i}{\partial y^2} & \cdots \\ \cdots & 2\frac{\partial^2 f_i}{\partial x \partial y} & 2\frac{\partial^2 g_i}{\partial x \partial y} & 2\frac{\partial^2 h_i}{\partial x \partial y} & \cdots \end{bmatrix}$$

3.2.3 Stress–Strain Relationships

Defining the stress vector $\underline{\sigma}$ as :

$$\underline{\sigma} = \{ \sigma_x \ \sigma_y \ \tau_{xy} \}$$

and using generalised Hooke's law, the stress vector $\underline{\sigma}$ can be expressed in terms of strain vector $\underline{\epsilon}$ as follows:

$$\underline{\sigma} = \underline{D} \, \underline{\epsilon}$$

where

$$\underline{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Hence

$$\underline{\sigma} = \underline{D} \, (\underline{\epsilon}_o - z \, \underline{\hat{\epsilon}}_b) \tag{3.11}$$

$$= \underline{\sigma}_o - z \, \underline{\hat{\sigma}}_b$$

where

$$\underline{\sigma}_o = \underline{D}_o \, \underline{\epsilon}_o = \underline{D} \, \underline{B}_o \, \underline{\delta}_o$$

$$\underline{\hat{\sigma}}_b = \underline{D} \, \underline{\hat{\epsilon}}_b = \underline{D} \, \underline{\hat{B}}_b \, \underline{\delta}_b$$

3.2.4 Derivation of element stiffness matrix

The strain energy density (strain energy per unit volume) is defined as follows:

$$\bar{U} = \frac{1}{2} \underline{\sigma}^t \, \underline{\epsilon} = \frac{1}{2} \underline{\epsilon}^t \, \sigma \tag{3.12}$$

from which the total strain energy of the element is expressed as follows:

$$U = \iiint \bar{U}\ dx\ dy\ dz$$

$$= \iint U'\ dx\ dy$$

where

$$U' = \int_{-h/2}^{h/2} \bar{U}\ dz \tag{3.13}$$

Hence, using equations (3.8) and (3.11), it can be deduced that:

$$U' = \frac{1}{2} \int_{-h/2}^{h/2} (\underline{\epsilon}_o^t - z\,\hat{\underline{\epsilon}}_b^t)\ \underline{D}\ (\underline{\epsilon}_o - z\,\hat{\underline{\epsilon}}_b)\ dz$$

therefore

$$U' = \frac{1}{2}\,(h\,\underline{\epsilon}_o^t\ \underline{D}\ \underline{\epsilon}_o + \frac{h^3}{12}\,\hat{\underline{\epsilon}}_b^t\ \underline{D}\ \hat{\underline{\epsilon}}_b)$$

Hence, the strain energy of the element can be expressed as follows:

$$U = \iint_{\substack{x-y \\ element}} U'\ dx\ dy$$

$$= \frac{1}{2} \iint_{element} [\ h\,\underline{\epsilon}_o^t\ \underline{D}\ \underline{\epsilon}_o + \frac{h^3}{12}\,\hat{\underline{\epsilon}}_b^t\ \underline{D}\ \hat{\underline{\epsilon}}_b\ ]\ dx\ dy$$

and from equations (3.9) and (3.10), it can be shown that

$$U = \frac{1}{2}\,\underline{\delta}_o^t\ \underline{K}_o\ \underline{\delta}_o + \frac{1}{2}\,\underline{\delta}_b^t\ \underline{K}_b\ \underline{\delta}_b \tag{3.14}$$

where

$$\underline{K}_o = \frac{1}{2} \iint_{element} h\ \underline{B}_o^t\ \underline{D}\ \underline{B}_o\ dx\ dy$$

$$\underline{K}_b = \frac{1}{2} \iint_{element} \frac{h^3}{12} \hat{\underline{B}}_b^t \, \underline{D} \, \hat{\underline{B}}_b \, dx \, dy$$

therefore

$$U = \frac{1}{2} \underline{\delta}^t \, \underline{K}_{(e)} \underline{\delta} \tag{3.15}$$

and the element stiffness matrix $\underline{K}_{(e)}$ is defined such that:

$$\underline{K}_{(e)} \, \underline{\delta} = \frac{\partial U}{\partial \underline{\delta}}$$

Hence, it can be proved that:

$$\underline{K}_{(e)} = \begin{bmatrix} \underline{K}_o & \underline{0} \\ \underline{0} & \underline{K}_b \end{bmatrix}$$

3.2.5 Derivation of elastic foundation stiffness matrix

For an n−noded plate−bending element resting on an elastic foundation, as shown in [Fig 3.3], the force acting on an infinitesimal element due to foundation effect is defined as follows:

$$\Delta F = - k_f \, w \, \Delta x \Delta y$$

where

k_f is known as the stiffness of foundation.

w is the displacement component in the z−direction.

The work done due to an infinitesimal deformation "δw" in the z−direction by the foundation force is as follows:

$$\delta \Delta W_f = - \delta w \, k_f \, w \, \Delta x \, \Delta y$$

i.e

$$\delta W_f = - \iint \delta w \, k_f \, w \, dx \, dy$$

and

$$W_f = - \frac{1}{2} \iint_{element} \underline{w}^t \, k_f \, \underline{w} \, dx \, dy$$

where

$$\underline{w} = [w]_{1\times 1}$$

Using equation (3.6), it can be shown that:

$$\underline{w} = \underline{N}_b \, \underline{\delta}_b$$

where

$$\underline{N}_b = [\ f_1 \quad g_1 \quad h_1 \qquad f_2 \ \ g_2 \ \ h_2 \qquad \cdots \quad]$$

Hence, it can be deduced that:

$$W_f = -\frac{1}{2} \underline{\delta}_b^t \, \underline{K}_f \, \underline{\delta}_b \tag{3.16}$$

where

$$\underline{K}_f = \iint_{element} k_f \, \underline{N}_b^t \, \underline{N}_b \, dx \, dy$$

Defining the equivalent nodal force vector "$\underline{F}_e$", such that the work done by applied external load is

$$W_e = \underline{\delta}^t \ \underline{F}_e \tag{3.17}$$

then the total potential energy of the element is

$$\chi = U - W$$

$$\chi = U - W_e - W_f \tag{3.18}$$

Substituting from equation (3.15), (3.16), (3.17), into (3.18), then:

$$\chi = \frac{1}{2} \underline{\delta}^t \, \underline{K}_{(e)} \, \underline{\delta} - \underline{\delta}^t \, \underline{F}_e + \frac{1}{2} \underline{\delta}_b^t \, \underline{K}_f \, \underline{\delta}_b$$

Using the minimum total potential energy theorem, it can be deduced that

$$\frac{\partial \chi}{\partial \underline{\delta}} = \underline{K}_{eq}\,\underline{\delta} - \underline{F}_e = \underline{0}$$

i.e.

$$\underline{K}_{eq}\,\underline{\delta} = \underline{F}_e$$

where

$$\underline{K}_{eq} = \begin{bmatrix} \underline{K}_o & \underline{0} \\ \underline{0} & \underline{K}_b + \underline{K}_f \end{bmatrix}$$

3.2.6 <u>Equivalent nodal force vector for a pressure loading</u>

Consider a case of a plate subjected to a pressure loading with intensity q(x,y) (load per unit area in the z direction), as shown in [Fig 3.4]. The force acting on an infinitesimal element due to the pressure loading is

$$\Delta F = q(x,y)\ \Delta x\ \Delta y$$

Hence,

$$W_e = \iint_{element} w\ q(x,y)\ dx\ dy$$

$$= \underline{\delta}_b^t \iint \underline{N}_b^t\ q(x,y)\ \ dx\ dy \equiv \underline{\delta}^t\ \underline{F}_e$$

Hence, the equivalent nodal force vector is:

$$\underline{F}_e = \begin{bmatrix} \underline{F}_o \\ \underline{F}_b \end{bmatrix}$$

where

$$\underline{F}_o = \underline{0},$$

$$\underline{F}_b = \iint_{element} \underline{N}_b^t \, q(x,y) \; dx \, dy$$

3.3 FIRST ORDER SHEAR ELEMENT

The basic assumptions for the derivation of Mindlin–plate elements are similar to those used for Kirchhoff element, except the effect of transverse shears is to be considered in order to improve the accuracy of such elements for the case of thick plates.

There are many thick plate elements available in the literature [Ref. 54]. In this section, an attempt will be made to derive a first–order shear element for thick plates on elastic foundation using a simplified approach.

3.3.1 Representation of transverse shears

Considering a flat plate of uniform thickness "h" with its mid-surface being in the x–y plane, as shown in fig 3.2, from the boundary conditions at upper and lower surfaces of the plate ($z=-\frac{h}{2}$, $z=+\frac{h}{2}$) it can be deduced that:

$$\tau_{xz} = \tau_{yz} = 0 \qquad\qquad \text{at } z = \pm\,\frac{h}{2}$$

If the values of transverse shear stresses are assumed at the mid-plane of the plate ($z=0$) as follows:

$$\tau_{xz} = \Phi(x,y)$$

$$\tau_{yz} = \Psi(x,y)$$

then for a homogeneous, isotropic material, distributions of τ_{xz} and τ_{yz} can be approximated over the z–axis (at any x,y), in terms of their values at

$$z_1 = -\,\frac{h}{2}, \qquad z_2 = 0, \qquad z_3 = +\frac{h}{2}$$

Hence, by using a 3−point Lagrangian interpolation, it can be deduced that:

$$\tau_{xz}(x,y,z) = \Phi(x,y)\,(1 - \tfrac{4z^2}{h^2}) \tag{3.19}$$

$$\tau_{yz}(x,y,z) = \Psi(x,y)(1 - \tfrac{4z^2}{h^2}) \tag{3.20}$$

and for a linearly elastic material:

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = \frac{\Phi(x,y)}{G}\,(1 - \tfrac{4z^2}{h^2}) \tag{3.21}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{\Psi(x,y)}{G}\,(1 - \tfrac{4z^2}{h^2}) \tag{3.22}$$

For the first−order shear element considered here, transverse shears are averaged in the z−direction, and the average values; $\overline{\tau}_{xz}$, $\overline{\tau}_{yz}$, $\overline{\gamma}_{xz}$ and $\overline{\gamma}_{yz}$ are functions of (x,y) only.

Average transverse shears, are assumed such that:

(i) The total shear forces (per unit length) over the thickness are the same, i.e at any point (x,y)

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\,dz = h\,\overline{\tau}_{xz} \tag{3.23}$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz}\,dz = h\,\overline{\tau}_{yz} \tag{3.24}$$

(ii) The contribution of the average transverse shears in the strain energy is the same as the actual transverse shears i.e

$$\tfrac{1}{2} \iiint (\tau_{xz}\,\gamma_{xz} + \tau_{yz}\,\gamma_{yz})\,dx\,dy\,dz$$

$$= \tfrac{1}{2} \iint h\,(\overline{\tau}_{xz}\,\overline{\gamma}_{xz} + \overline{\tau}_{yz}\,\overline{\gamma}_{yz})\,dx\,dy$$

or simply

$$\int_{-h/2}^{h/2} \tau_{xz}\, \gamma_{xz}\, dz = h\, \overline{\tau}_{xz}\, \overline{\gamma}_{xz} \tag{3.25}$$

$$\int_{-h/2}^{h/2} \tau_{yz}\, \gamma_{yz}\, dz = h\, \overline{\tau}_{yz}\, \overline{\gamma}_{yz} \tag{3.26}$$

Substituting from equation (3.19), (3.20) into (3.23) and (3.24), it can be shown that

$$\Phi(x,y) = \tfrac{3}{2}\, \overline{\tau}_{xz} \tag{3.27}$$

and similarly,

$$\Psi(x,y) = \tfrac{3}{2}\, \overline{\tau}_{yz} \tag{3.28}$$

Substituting from equations (3.19) to (3.22) into (3.25) and (3.26), it can be shown that

$$h\, \overline{\tau}_{xz}\, \overline{\gamma}_{xz} = \int_{-h/2}^{h/2} \frac{\Phi^2}{G}\, (1 + \frac{4z^2}{h^2})^2 \; dz$$

$$= \frac{8\Phi^2}{15\ G}$$

i.e.

$$\overline{\tau}_{xz}\, \overline{\gamma}_{xz} = \frac{8\Phi^2}{15\ G} \tag{3.29}$$

and similarly,

$$\overline{\tau}_{yz}\, \overline{\gamma}_{yz} = \frac{8\ \Psi^2}{15\ G} \tag{3.30}$$

Substituting from equations (3.27), (3.28) into (3.29) and (3.30), it can be shown that

$$\overline{\tau}_{xz}\, \overline{\gamma}_{xz} = \frac{8}{15G}\, \frac{9(\tau_{xz})^2}{4}$$

i.e.

$$\overline{\gamma}_{xz} = \frac{\overline{\tau}_{xz}}{\left(\frac{5}{6}G\right)} \tag{3.31}$$

and similarly,

$$\overline{\gamma}_{yz} = \frac{\overline{\tau}_{yz}}{\left(\frac{5}{6}G\right)} \tag{3.32}$$

Hence, equilibrium and strain energy are obtained by using average transverse shears, such that, equations, (3.31), (3.32) are valid.

3.3.2 Displacement Components

For the first order shear element presented here, the average transverse shears are used instead of the actual values, i.e. the transverse shears are considered functions of x−y only. Using the usual Cauchy's strain−displacement relationships, it can be deduced that:

$$\overline{\gamma}_{xz}(x,y) \simeq \frac{\partial u}{\partial z} + \frac{\partial w(x,y)}{\partial x}$$

$$\overline{\gamma}_{yz}(x,y) \simeq \frac{\partial v}{\partial z} + \frac{\partial w(x,y)}{\partial y}$$

i.e.

$$\frac{\partial u}{\partial z} \simeq - \frac{\partial w(x,y)}{\partial x} + \overline{\gamma}_{xz}(x,y)$$

and

$$\frac{\partial v}{\partial z} \simeq - \frac{\partial w(x,y)}{\partial y} + \overline{\gamma}_{yz}(x,y)$$

Hence by integration with respect to "z", it can be shown that

$$u(x,y,z) \simeq u_o(x,y) - z\left(\frac{\partial w}{\partial x} - \overline{\gamma}_{xz} \right)$$

$$v(x,y,z) \simeq v_o(x,y) - z\left(\frac{\partial w}{\partial y} - \overline{\gamma}_{yz} \right)$$

where $u_o(x,y)$ and $v_o(x,y)$ are the displacement components at z=0, in the x and y directions respectively.

Engineering slope angles θ_x, θ_y can be defined such that

$$\theta_x(x,y) = \frac{\partial w}{\partial y} - \bar{\gamma}_{yz} \tag{3.33a}$$

and

$$\theta_y(x,y) = -\left(\frac{\partial w}{\partial x} - \bar{\gamma}_{xz} \right) \tag{3.33b}$$

Hence it can be shown that,

$$u(x,y,z) \simeq u_o(x,y) + z\,\theta_y(x,y) \tag{3.34a}$$

$$v(x,y,z) \simeq v_o(x,y) - z\,\theta_x(x,y) \tag{3.34b}$$

$$w(x,y,z) \simeq w(x,y) \tag{3.34c}$$

which would result if we assume that any normal to the mid-plane before deformation remains straight but not necessarily normal after deformation [Ref 54], as shown in Fig 3.5.

In order to obtain continuous displacement components all over the plate, the nodal displacement vector for an n−noded element is to be defined in terms of nodal values of u_o, v_o, w, θ_x and θ_y, i.e

$$\underline{\delta} = \begin{bmatrix} \underline{\delta}_o \\ \underline{\delta}_b \end{bmatrix} \tag{3.35}$$

where

$$\underline{\delta}_o = \begin{bmatrix} (u_o)_1 \\ (v_o)_1 \\ \vdots \\ (u_o)_n \\ (v_o)_n \end{bmatrix}$$

and

$$\underline{\delta}_b = \begin{bmatrix} w_1 \\ (\theta_x)_1 \\ (\theta_y)_1 \\ \vdots \\ w_n \\ (\theta_x)_n \\ (\theta_y)_n \end{bmatrix}$$

and it is clear from equations (3.33), that θ_x and θ_y are independent of w for a thick plate, i.e. Lagrangian interpolation could be employed such that

$$u_o(x,y) = \sum_{i=1}^{n} (u_o)_i \; N_i(x,y) \tag{3.36a}$$

$$v_o(x,y) = \sum_{i=1}^{n} (v_o)_i \; N_i(x,y) \tag{3.36b}$$

$$w(x,y) = \sum_{i=1}^{n} w_i \; N_i(x,y) \tag{3.36c}$$

$$\theta_x(x,y) = \sum_{i=1}^{n} (\theta_x)_i \; N_i(x,y) \tag{3.36d}$$

$$\theta_y(x,y) = \sum_{i=1}^{n} (\theta_y)_i \; N_i(x,y) \tag{3.36e}$$

where

$(u_o)_i$, $(v_o)_i$, w_i	are displacement components at the $i-th$ node on the neutral plane in the x, y, z directions.
θ_x, θ_y	are the slope angles with respect to x, y.

3.3.3 Strain Components

Using Cauchy strain–displacement relationships together with equations (3.34), it can be shown that:

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} + z\,\frac{\partial \theta_y}{\partial x}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} - z\,\frac{\partial \theta_x}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} - z\left(\frac{\partial \theta_x}{\partial x} - \frac{\partial \theta_y}{\partial y} \right)$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\theta_x + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \theta_y + \frac{\partial w}{\partial x}$$

Strain vectors are defined such that:

$$\underline{\epsilon}_{xy} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \underline{\epsilon}_o + \underline{\epsilon}_b \tag{3.37}$$

and

$$\underline{\epsilon}_s = \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} -\theta_x + \frac{\partial w}{\partial y} \\ \theta_y + \frac{\partial w}{\partial x} \end{bmatrix} \tag{3.38}$$

where

$$\underline{\epsilon}_o = \begin{bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial v_o}{\partial y} \\ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \end{bmatrix} \tag{3.39}$$

$$\underline{\epsilon}_b = -z \begin{bmatrix} -\frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_x}{\partial x} - \frac{\partial \theta_y}{\partial y} \end{bmatrix} = -z\, \underline{\hat{\epsilon}}_b \tag{3.40}$$

Substituting equations (3.36) into (3.38), (3.39) and (3.40) it can be proved that:

$$\underline{\epsilon}_o = \underline{B}_o \, \underline{\delta}_o \tag{3.41}$$

$$\underline{\epsilon}_b = -z\, \hat{\epsilon}_b = -z\, \underline{\hat{B}}_b \, \underline{\delta}_b \tag{3.42}$$

$$\underline{\epsilon}_s = \underline{B}_s \, \underline{\delta}_b \tag{3.43}$$

where $\underline{B}_o$ is as defined in section 3.2.2, and

$$\underline{\hat{B}}_b = \begin{bmatrix} \cdots & 0 & 0 & -\frac{\partial N_i}{\partial x} & \cdots \\ \cdots & 0 & \frac{\partial N_i}{\partial y} & 0 & \cdots \\ \cdots & 0 & \frac{\partial N_i}{\partial x} & -\frac{\partial N_i}{\partial y} & \cdots \end{bmatrix} \tag{3.44}$$

$$\underline{B}_s = \begin{bmatrix} \cdots & \frac{\partial N_i}{\partial y} & -N_i & 0 & \cdots \\ \cdots & \frac{\partial N_i}{\partial x} & 0 & N_i & \cdots \end{bmatrix} \tag{3.45}$$

3.3.4 Stress Components

Using stress strain relationships for a plane–stress case with linearly elastic material, it can be shown that:

(i) $$\underline{\sigma}_{xy} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \underline{D} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \underline{\sigma}_o + \underline{\sigma}_b \tag{3.46}$$

where

$$\underline{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\underline{\sigma}_o = \underline{D}\,\underline{\epsilon}_o = \underline{D}\,\underline{B}_o\,\underline{\delta}_o \tag{3.47}$$

$$\underline{\sigma}_b = \underline{D}\,\underline{\epsilon}_b = -\,z\,\underline{D}\,\underline{\hat{B}}_b\,\underline{\delta}_b \tag{3.48}$$

(ii) $$\underline{\sigma}_s = \begin{bmatrix} \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \underline{D}_s \begin{bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

$$= \underline{D}_s\,\underline{B}_s\,\underline{\delta}_b \tag{3.49}$$

where

$$\underline{D}_s = \frac{5}{6}\begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}.$$

3.3.4 Derivation of Element Stiffness matrix

The strain energy of the element can be defined as follows:

$$U = \iiint \bar{U}\ dx\ dy\ dz = \iint U'\ dx\ dy \tag{3.50}$$

where

$$\bar{U} = \text{strain energy density}$$

$$= \frac{1}{2}\underline{\sigma}^t\,\underline{\epsilon}$$

and

$$U' = \int_{-h/2}^{h/2} \bar{U}\ dz \tag{3.51}$$

From which it can be shown:

$$\bar{U} = \tfrac{1}{2}\left(\sigma_x\ \epsilon_x + \sigma_y\ \epsilon_y + \cdots\ \tau_{zx}\ \gamma_{zx}\right)$$

$$= \bar{U}_{xy} + \bar{U}_s$$

where

$$\bar{U}_{xy} = \tfrac{1}{2}\,\underline{\sigma}^t_{xy}\,\underline{\epsilon}_{xy}$$

and

$$\bar{U}_s = \tfrac{1}{2}\,\underline{\sigma}^t_s\,\underline{\epsilon}_s$$

i.e. $\qquad \bar{U}_{xy} = \tfrac{1}{2}\left(\underline{\epsilon}_o - z\,\hat{\underline{\epsilon}}_b\right)^t \underline{D}\left(\underline{\epsilon}_o - z\,\hat{\underline{\epsilon}}_b\right)$

and

$$\bar{U}_s = \tfrac{1}{2}\,\underline{\epsilon}_s^t\,\underline{D}_s\,\underline{\epsilon}_s$$

Hence, by integration with respect to "z", it can be proved that

$$U' = U'_o + U'_b + U'_s$$

where

$$U'_o = \frac{h}{2}\,\underline{\epsilon}_o^t\,\underline{D}\,\underline{\epsilon}_o \tag{3.52}$$

$$U'_b = \frac{1}{2}\left(\frac{h^3}{12}\right)\hat{\underline{\epsilon}}_b^t\,\underline{D}\,\hat{\underline{\epsilon}}_b \tag{3.53}$$

$$U'_s = \frac{h}{2}\,\underline{\epsilon}_s^t\,\underline{D}_s\,\underline{\epsilon}_s \tag{3.54}$$

Substituting from equations (3.41), (3.42) and (3.43) into equations (3.52), (3.53) and (3.54), and using the element strain energy given by equation (3.50), it can be shown that:

$$U = \frac{1}{2}\,\underline{\delta}_o^t\,\underline{K}_o\,\underline{\delta}_o + \frac{1}{2}\,\underline{\delta}_b^t\,\underline{K}_b\,\underline{\delta}_b + \frac{1}{2}\,\underline{\delta}_b^t\,\underline{K}_s\,\underline{\delta}_b \tag{3.55}$$

where

$$\underline{K}_o = \iint_{element} h\,\underline{B}_o^t\,\underline{D}\,\underline{B}_o\,dx\,dy \tag{3.56}$$

$$\underline{K}_b = \iint_{element} \frac{h^3}{12}\,\hat{\underline{B}}_b^t\,\underline{D}\,\hat{\underline{B}}_b\,dx\,dy \tag{3.57}$$

$$\underline{K}_s = \iint_{element} h\,\underline{B}_s^t\,\underline{D}_s\,\underline{B}_s\,dx\,dy \tag{3.58}$$

3.3.5 Derivation of Elastic Foundation Stiffness matrix

Using an analysis similar to that given in section 3.2.5, it is clear that the work done due to foundation stiffness, and a vertical deformation of the plate w(x,y) can be expressed as follows:

$$W_f = -\frac{1}{2}\iint_{element} \underline{w}^t\, k_f\, \underline{w}\ dx\ dy \tag{3.59}$$

where

$$\underline{w} = \underline{N}_b\, \underline{\delta}_b \tag{3.60}$$

and

$$\underline{N}_b = [\ \cdots\ N_i\ \ 0\ \ 0\ \ \cdots\ ] \tag{3.61}$$

Hence it can be deduced that

$$W_f = -\frac{1}{2}\ \underline{\delta}_b^t\ \underline{K}_f\ \underline{\delta}_b \tag{3.62}$$

where

$$\underline{K}_f = \iint_{element} k_f\ \underline{N}_b^t\, \underline{N}_b\ dx\ dy \tag{3.63}$$

From the definition of $\underline{N}_b$ it can also be proved that:

$$\underline{K}_f\ [\ 3(i-1)+r\ ,\ 3\ (j-1)+s\ ]$$

$$=0 \qquad \text{if } r\neq 1 \text{ or } s\neq 1$$

$$=\iint_{element} k_f\ N_i\ N_j\ dx\ dy \qquad \text{for } r=s=1 \tag{3.64}$$

If "$\underline{F}_e$" represents the nodal force vector equivalent to applied forces, then the total potential energy of the element can be expressed as follows:

$$\chi = U - W_f - W_e$$

$$= \frac{1}{2}\underline{\delta}_o^t\ \underline{K}_o\ \underline{\delta}_o + \frac{1}{2}\underline{\delta}_b^t\left(\ \underline{K}_b + \underline{K}_s + \underline{K}_f\ \right)\underline{\delta}_b - \underline{\delta}^t\ \underline{F}_e \tag{3.65}$$

Defining "K_{eq}" as the total stiffness matrix of the element such that:

$$\frac{\partial \chi}{\partial \underline{\delta}} = \underline{K}_{eq}\,\underline{\delta} - \underline{F}_e = \underline{0}$$

i.e.

$$\underline{K}_{eq}\,\underline{\delta} = \underline{F}_e$$

it can be shown that

$$\underline{K}_{eq} = \begin{bmatrix} \underline{K}_o & \underline{0} \\ \underline{0} & \underline{K}_b+\underline{K}_s+\underline{K}_f \end{bmatrix}.$$

The method for the derivation of "$\underline{K}_{eq}$" as explained above, will lead to separate calculations of $\underline{K}_o$, $\underline{K}_b$, $\underline{K}_s$ and $\underline{K}_f$. Hence, to allow the element to be safely used for thin plates, a reduced integration can be invoked by using a Gaussian quadrature scheme for the numerical evaluation of $\underline{K}_s$ of an order less than that of $\underline{K}_b$. For an 8−node or 9−node quadrilateral element, a 2x2 scheme is suggested for $\underline{K}_s$ whilst a 3x3 scheme is employed for $\underline{K}_b$, and any one of the two schemes can be used for $\underline{K}_o$ and $\underline{K}_f$. Nodal force vectors equivalent to pressure loading can be estimated in a way similar to that explained in section 3.2.6.

3.4 HIGH-ORDER FACET-SHELL ELEMENT

Although the previous first−order element has a simple derivation and it is based upon Lagrangian shape functions, it is not very accurate for thin plates and even with reduced integration, one should use elements with mid-side nodes (such as the 9−node quadrilateral element) in order to obtain accuracy comparable with that obtained by means of Kirchhoff's elements. In this section, an attempt is made to derive a high order shear facet−shell element for thin−thick plates on an elastic foundation using a simplified displacement approach.

3.4.1 Displacement Components

Starting with parabolic transverse shear strain distribution as given by means of equation (3.21) and (3.22), the following approximate equations can be assumed:

$$\gamma_{xz}(x,y,z) = \gamma^{o}_{xz}(x,y)\left[\,1 - \frac{4z^2}{h^2}\,\right] \qquad (3.66)$$

$$\gamma_{yz}(x,y,z) = \gamma^o_{yz}(x,y)\left[1 - \frac{4z^2}{h^2}\right] \qquad (3.67)$$

where γ^o_{xz} and γ^o_{yz} are the values of transverse shear strains at the mid-plane of the plate (z=0). Using Cauchy's strain displacement relationships, then it can be shown that:

$$\gamma_{xz}(x,y,z) = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \left[1 - \frac{4z^2}{h^2}\right]\gamma^o_{xz}$$

$$\gamma_{yz}(x,y,z) = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \left[1 - \frac{4z^2}{h^2}\right]\gamma^o_{yz}$$

i.e.

$$\frac{\partial u}{\partial z} = -\frac{\partial w(x,y)}{\partial x} + \left(1 - \frac{4z^2}{h^2}\right)\gamma^o_{xz}(x,y) \qquad (3.68)$$

$$\frac{\partial v}{\partial z} = -\frac{\partial w(x,y)}{\partial y} + \left(1 - \frac{4z^2}{h^2}\right)\gamma^o_{yz}(x,y) \qquad (3.69)$$

Integrating equations (3.68), (3.69) with respect to z, it can be proved that:

$$u(x,y,z) \simeq u_o(x,y) - z\frac{\partial w}{\partial x} + \left(z - \frac{4z^3}{3h^2}\right)\gamma^o_{xz} \qquad (3.70a)$$

$$v(x,y,z) \simeq v_o(x,y) - z\frac{\partial w}{\partial y} + \left(z - \frac{4z^3}{3h^2}\right)\gamma^o_{yz} \qquad (3.70b)$$

where

$$w(x,y,z) \simeq w(x,y) \qquad (3.70c)$$

In order to have u, v, and w continuous all over the plate, they should be interpolated in terms of the plate nodal values defined for an n–noded element by means of the following vector.

$$\underline{\delta} = \left\{\underline{\delta}_o \;\; \underline{\delta}_b \;\; \underline{\delta}_s\right\}$$

where

$$\underline{\delta}_o = \left\{(u_o)_1 \; (v_o)_1 \; \cdots \; (u_o)_n \; (v_o)_n\right\} \qquad (3.71)$$

$$\underline{\delta}_b = \left\{w_1 \; w_{1,x} \; w_{1,y} \; \cdots \; w_n \; w_{n,x} \; w_{n,y}\right\} \qquad (3.72)$$

$$\underline{\delta}_s = \left\{(\gamma^o_{xz})_1 \; (\gamma^o_{yz})_1 \; \cdots \; (\gamma^o_{xz})_n \; (\gamma^o_{yz})_n\right\} \qquad (3.73)$$

The values of $u_o(x,y)$, $v_o(x,y)$, $\gamma^o_{xz}(x,y)$, $\gamma^o_{yz}(x,y)$ can be interpolated in terms of their nodal values and Lagrangian shape functions as follows:

$$u_o(x,y) = \sum_{i=1}^{n} (u_o)_i \; N_i(x,y) \tag{3.74a}$$

$$v_o(x,y) = \sum_{i=1}^{n} (v_o)_i \; N_i(x,y) \tag{3.74b}$$

$$\gamma^o_{xz}(x,y) = \sum_{i=1}^{n} (\gamma^o_{xz})_i \; N_i(x,y) \tag{3.74c}$$

$$\gamma^o_{yz}(x,y) = \sum_{i=1}^{n} (\gamma^o_{yz})_i \; N_i(x,y) \tag{3.74d}$$

However due to the presence of nodal derivatives of w, the interpolated function of w should be represented in terms of Hermitian shape functions, as given by equation (3.6), i.e.

$$w(x,y) = \sum_{i=1}^{n} [\, w_i \, f_i(x,y) + w_{i,x} \, g_i(x,y) + w_{i,y} \, h_i(x,y)] \tag{3.75}$$

3.4.2 Strain Components

Using Cauchy strain–displacement relationships, the strain components at any point (x,y,z) inside the element can be expressed as follows:

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2} + \frac{\partial \gamma^o_{xz}}{\partial x}\left(z - \frac{4z^3}{3h^2} \right) \tag{3.76a}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} - z\,\frac{\partial^2 w}{\partial y^2} + \frac{\partial \gamma^o_{yz}}{\partial y}\left(z - \frac{4z^3}{3h^2} \right) \tag{3.76b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \right) - 2\,z\,\frac{\partial^2 w}{\partial x \partial y} + \left(\frac{\partial \gamma^o_{xz}}{\partial y} + \frac{\partial \gamma^o_{yz}}{\partial x} \right)\left(z - \frac{4z^3}{3h^2} \right) \tag{3.76c}$$

$$\gamma_{xz} = \gamma^o_{xz}\left(1 - \frac{4z^2}{h^2} \right) \tag{3.77a}$$

$$\gamma_{yz} = \gamma^o_{yz}\left(1 - \frac{4z^2}{h^2} \right) \tag{3.77b}$$

for simplification of the derivation the strain vector will be partitioned in terms of the following 2 vectors

$$\underline{\epsilon}_{xy} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xz} \end{bmatrix} \tag{3.78}$$

$$\underline{\epsilon}_s = \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{3.79}$$

Defining the following vectors:

$$\underline{\epsilon}_o = \begin{bmatrix} \frac{\partial u_o}{\partial x} \\ \frac{\partial v_o}{\partial y} \\ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \end{bmatrix} \tag{3.80}$$

and

$$\hat{\underline{\epsilon}}_s = \begin{bmatrix} \gamma^o_{xz}(x,y) \\ \gamma^o_{yz}(x,y) \end{bmatrix} \tag{3.81}$$

$$\underline{\epsilon}_t = \begin{bmatrix} \frac{\partial \gamma^o_{xz}}{\partial x} \\ \frac{\partial \gamma^o_{yz}}{\partial y} \\ \frac{\partial \gamma^o_{xz}}{\partial y} + \frac{\partial \gamma^o_{yz}}{\partial x} \end{bmatrix} \tag{3.82}$$

$$\hat{\underline{\epsilon}}_b = \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\,\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{3.83}$$

then it can be deduced that: ·

$$\underline{\epsilon}_{xy} = \underline{\epsilon}_o - z\,\hat{\underline{\epsilon}}_b + \left(z - \frac{4z^3}{3h^2} \right) \underline{\epsilon}_t \tag{3.84}$$

and

$$\underline{\epsilon}_s = \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \left(1 - \frac{4z^2}{h^2} \right) \hat{\underline{\epsilon}}_s \tag{3.85}$$

Using the interpolation functions as given by equations (3.74) and (3.75), it can be proved that:

$$\underline{\epsilon}_o = \underline{B}_o\, \underline{\delta}_o \tag{3.86a}$$

$$\underline{\epsilon}_t = \underline{B}_o\, \underline{\delta}_s \tag{3.86b}$$

$$\hat{\underline{\epsilon}}_s = \underline{B}_s\, \underline{\delta}_s \tag{3.86c}$$

$$\hat{\underline{\epsilon}}_b = \underline{B}_b\, \underline{\delta}_b \tag{3.86d}$$

where

$$\underline{B}_o = \begin{bmatrix} \cdots & \frac{\partial N_i}{\partial x} & 0 & \cdots \\ \cdots & 0 & \frac{\partial N_i}{\partial y} & \cdots \\ \cdots & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \cdots \end{bmatrix} \tag{3.87}$$

and

$$\underline{B}_s = \begin{bmatrix} \cdots & N_i & 0 & \cdots \\ \cdots & 0 & N_i & \cdots \end{bmatrix} \tag{3.88}$$

$$\underline{B}_b = \begin{bmatrix} \cdots & \frac{\partial^2 f_i}{\partial x^2} & \frac{\partial^2 g_i}{\partial x^2} & \frac{\partial^2 h_i}{\partial x^2} & \cdots \\ \cdots & \frac{\partial^2 f_i}{\partial y^2} & \frac{\partial^2 g_i}{\partial y^2} & \frac{\partial^2 h_i}{\partial y^2} & \cdots \\ \cdots & 2\,\frac{\partial^2 f_i}{\partial x \partial y} & 2\,\frac{\partial^2 g_i}{\partial x \partial y} & 2\,\frac{\partial^2 h_i}{\partial x \partial y} & \cdots \end{bmatrix} \qquad (3.89)$$

3.4.3 Stress Components

The stress components at any point can be defined in terms of the partitioned vectors $\underline{\sigma}_{xy}$, $\underline{\sigma}_s$ where

$$\underline{\sigma}_{xy} = \left\{ \sigma_x \quad \sigma_y \quad \tau_{xy} \right\} \qquad (3.90)$$

and

$$\underline{\sigma}_s = \left\{ \tau_{xy} \quad \tau_{yz} \right\} \qquad (3.91)$$

using generalised Hooke's law it can be deduced that

$$\underline{\sigma}_{xy} = \underline{D}\,\underline{\epsilon}_{xy}$$

$$\underline{\sigma}_s = \underline{D}_s\,\underline{\epsilon}_s$$

where

$$\underline{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

and

$$\underline{D}_s = G \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$G = \frac{E}{2(1+\nu)}$$

Using equations (3.84), (3.85) and (3.86) the stress vectors can be expressed in terms of nodal displacements and shape functions as follows

$$\underline{\sigma}_{xy} = \underline{D}\left[\,\underline{\epsilon}_o - z\,\underline{\hat{\epsilon}}_b + \left(z - \frac{4z^3}{3h^2}\right)\underline{\epsilon}_t\,\right]$$

$$= \underline{D}\left[\,\underline{B}_o\,\underline{\delta}_o - z\,\underline{B}_b\,\underline{\delta}_b + \left(z - \frac{4z^3}{3h^2}\right)\underline{B}_o\,\underline{\delta}_s\,\right] \tag{3.92}$$

and

$$\underline{\sigma}_s = \underline{D}_s\left(1 - \frac{4z^2}{h^2}\right)\underline{\hat{\epsilon}}_s$$

$$= \underline{D}_s\left(1 - \frac{4z^2}{h^2}\right)\underline{B}_s\,\underline{\delta}_s \tag{3.93}$$

3.4.4 Element Strain Energy

The strain energy per unit volume can be expressed at any point (x,y,z) as follows:

$$\bar{U} = \frac{1}{2}\left(\sigma_x\,\epsilon_x + \sigma_y\,\epsilon_y + \sigma_z\,\epsilon_z + \tau_{xy}\,\gamma_{xy} + \tau_{yz}\,\gamma_{yz} + \tau_{zx}\,\gamma_{zx}\right)$$

$$= \frac{1}{2}\,\underline{\sigma}^t_{xy}\,\underline{\epsilon}_{xy} + \frac{1}{2}\,\underline{\sigma}^t_s\,\underline{\epsilon}_s$$

$$= \frac{1}{2}\left[\underline{\delta}^t_o\,\underline{B}^t_o - z\,\underline{\delta}^t_b\,\underline{B}^t_b + \left(z-\frac{4z^3}{3h^2}\right)\underline{\delta}^t_s\,\underline{B}^t_o\right]\underline{D}\left[\underline{B}_o\,\underline{\delta}_o - z\,\underline{B}_b\,\underline{\delta}_b + \left(z-\frac{4z^3}{3h^2}\right)\underline{B}_o\,\underline{\delta}_s\right]$$

$$+ \frac{1}{2}\left(1-\frac{4z^2}{h^2}\right)^2\underline{\delta}^t_s\,\underline{B}^t_s\,\underline{D}_s\,\underline{B}_s\,\underline{\delta}_s$$

and the total strain is

$$U = \iiint_{-h/2}^{h/2} \bar{U}\,dz\,dx\,dy = \iint_{element} U'\,dx\,dy$$

where

$$U' = \int_{-h/2}^{h/2} \bar{U}\,dz$$

Integrating $\bar{U}$ with respect to z and using the following properties:

$$\int_{-h/2}^{h/2} z^{2n+1}\, dz = 0$$

$$\int_{-h/2}^{h/2} z^{2n}\, dz = \frac{2}{(2n+1)} \left(\frac{h}{2} \right)^{2n+1}$$

it can be proved that

$$U' = U'_o + U'_b + U'_t - U'_{bt} - U'_{tb} + U'_s \tag{3.94}$$

where

$$U'_o = \tfrac{1}{2}\, \underline{\delta}^t_o\, \underline{B}^t_o\, \underline{D}_o\, \underline{B}_o\, \underline{\delta}_o$$

$$U'_b = \tfrac{1}{2}\, \underline{\delta}^t_b\, \underline{B}^t_b\, \underline{D}_b\, \underline{B}_b\, \underline{\delta}_b$$

$$U'_t = \tfrac{1}{2}\, \underline{\delta}^t_s\, \underline{B}^t_o\, \underline{D}_t\, \underline{B}_o\, \underline{\delta}_s$$

$$U'_s = \tfrac{1}{2}\, \underline{\delta}^t_s\, \underline{B}^t_s\, \underline{D}_s\, \underline{B}_s\, \underline{\delta}_s$$

$$U'_{bt} = \tfrac{1}{2}\, \underline{\delta}^t_b\, \underline{B}^t_b\, \underline{D}_{bt}\, \underline{B}_o\, \underline{\delta}_s$$

$$U'_{tb} = \tfrac{1}{2}\, \underline{\delta}^t_s\, \underline{B}^t_o\, \underline{D}_{tb}\, \underline{B}_b\, \underline{\delta}_b$$

and the modified $\underline{D}$ matrices are defined as follows

$$\underline{D}_o = \int_{-h/2}^{h/2} \underline{D}\, dz = h\underline{D}$$

$$\underline{D}_b = \int_{-h/2}^{h/2} z^2\, \underline{D}\, dz = \frac{h^3}{12}\underline{D}$$

$$\underline{D}_t = \int_{-h/2}^{h/2} \left(z - \frac{4z^3}{3h^2}\right)^2 \underline{D}\, dz = \frac{17}{315}\, h^3\, \underline{D}$$

$$\underline{D}_s = \int_{-h/2}^{h/2} (1 - \frac{4z^2}{h^2})\ \underline{D}_s\ dz = \frac{8}{15}\ h\ \underline{D}_s$$

$$\underline{D}_{bt} = \int_{-h/2}^{h/2} (z^2 - \frac{4z^4}{3h^2})\ \underline{D}\ dz = \frac{1}{15}\ h^3\ \underline{D}$$

Hence, the total strain energy of the element can be expressed as follows:

$$U = \iint_{element} U'\ dx\ dy$$

$$U = U_o + U_b + U_t - U_{bt} - U_{tb} + U_s \tag{3.95}$$

where

$$U_\alpha = \iint U'_\alpha\ dx\ dy$$

$$\alpha \equiv 0,\ b,\ t,\ s,\ bt,\ tb$$

3.4.5 Element Stiffness Matrix

From the previous definition of element strain energy, partitioned stiffness matrices for the element can be defined from the following equivalent relationships.

$$U_o = \tfrac{1}{2}\ \underline{\delta}_o^t\ \underline{K}_o\ \underline{\delta}_o \equiv \iint U'_o\ dx\ dy \tag{3.96a}$$

$$U_b = \tfrac{1}{2}\ \underline{\delta}_b^t\ \underline{K}_b\ \underline{\delta}_b \equiv \iint U'_b\ dx\ dy \tag{3.96b}$$

$$U_t = \tfrac{1}{2}\ \underline{\delta}_s^t\ \underline{K}_t\ \underline{\delta}_s \equiv \iint U'_t\ dx\ dy \tag{3.96c}$$

$$U_s = \tfrac{1}{2}\ \underline{\delta}_s^t\ \underline{K}_s\ \underline{\delta}_s \equiv \iint U'_s\ dx\ dy \tag{3.96d}$$

$$U_{bt} = \tfrac{1}{2}\ \underline{\delta}_b^t\ \underline{K}_{bt}\ \underline{\delta}_s \equiv \iint U'_{bt}\ dx\ dy \tag{3.96e}$$

$$U_{tb} = \tfrac{1}{2}\ \underline{\delta}_s^t\ \underline{K}_{tb}\ \underline{\delta}_b \equiv \iint U'_{tb}\ dx\ dy \tag{3.96f}$$

Hence, the partitioned stiffness matrices can be expressed as follows:

$$\underline{\underline{K}}_o = \iint \underline{\underline{B}}_o^t \, \underline{\underline{D}}_o \, \underline{\underline{B}}_o \, dx\, dy$$

$$\underline{\underline{K}}_b = \iint \underline{\underline{B}}_b^t \, \underline{\underline{D}}_b \, \underline{\underline{B}}_b \, dx\, dy$$

$$\underline{\underline{K}}_t = \iint \underline{\underline{B}}_o^t \, \underline{\underline{D}}_t \, \underline{\underline{B}}_o \, dx\, dy$$

$$\underline{\underline{K}}_s = \iint \underline{\underline{B}}_s^t \, \underline{\underline{D}}_s \, \underline{\underline{B}}_s \, dx\, dy$$

$$\underline{\underline{K}}_{bt} = \iint \underline{\underline{B}}_b^t \, \underline{\underline{D}}_{bt} \, \underline{\underline{B}}_o \, dx\, dy$$

and

$$\underline{\underline{K}}_{tb} = \underline{\underline{K}}_{bt}^t$$

3.4.6 Elastic Foundation Stiffness Matrix

Equation (3.75) can be expressed in matrix form as follows:

$$w(x,y) = \underline{\underline{N}}_b \, \underline{\delta}_b$$

where

$$\underline{\underline{N}}_b = [\cdots \; f_i \;\; g_i \;\; h_i \;\; \cdots]_{1\times 3n}$$

The generated force due to the foundation elasticity can be obtained at an infinitesimal element of area [Fig 3.3] as follows:

$$\Delta F = -\, k_f \, w \, \Delta x \Delta y$$

Hence the work done by the force is

$$\Delta W_f = \int_0^w \Delta F \, dw$$

$$= \int_0^w (-\, k_f \, w) \, dw \, \Delta x \, \Delta y$$

$$= -\frac{1}{2} k_f \, w^2 \, \Delta x \, \Delta y$$

i.e

$$W_f = -\frac{1}{2} \iint_{element} k_f \, w^2 \, dx \, dy$$

$$= -\frac{1}{2} \iint_{element} k_f \, \underline{\delta}_b^t \, \underline{N}_b^t \, \underline{N}_b \, \underline{\delta}_b \, dx \, dy$$

$$= -\frac{1}{2} \underline{\delta}_b^t \, \underline{K}_f \, \underline{\delta}_b \tag{3.97}$$

where

$$\underline{K}_f = \iint_{element} k_f \, \underline{N}_b^t \, \underline{N}_b \, dx \, dy$$

3.4.7 Equivalent Total Element Stiffness Matrix

Defining $F_{(e)}$ as the nodal force vector equivalent to the actual loads, such that the work done by the actual loads can be expressed as follows:

$$W_e = \underline{\delta}^t \, F_e \tag{3.98}$$

then the total potential energy of the element can be expressed as follows:

$$\chi = U - W_f - W_e \tag{3.99}$$

Substituting from equations (3.95), (3.96), (3.97), (3.98) into (3.99), then it can be shown that:

$$\chi = \frac{1}{2} \underline{\delta}_o^t \, \underline{K}_o \, \underline{\delta}_o + \frac{1}{2} \underline{\delta}_b^t \, (\underline{K}_b + \underline{K}_f) \, \underline{\delta}_b + \frac{1}{2} \underline{\delta}_s^t \, (\underline{K}_s + \underline{K}_t) \, \underline{\delta}_s - \frac{1}{2} \underline{\delta}_b^t \, \underline{K}_{bt} \, \underline{\delta}_s$$

$$-\frac{1}{2} \underline{\delta}_s^t \, \underline{K}_{tb} \, \underline{\delta}_b - \underline{\delta}^t F_{(e)} = minimum \tag{3.100}$$

From the minimum total potential energy theorem, a total stiffness matrix equivalent to the partitioned matrices, can be defined such that

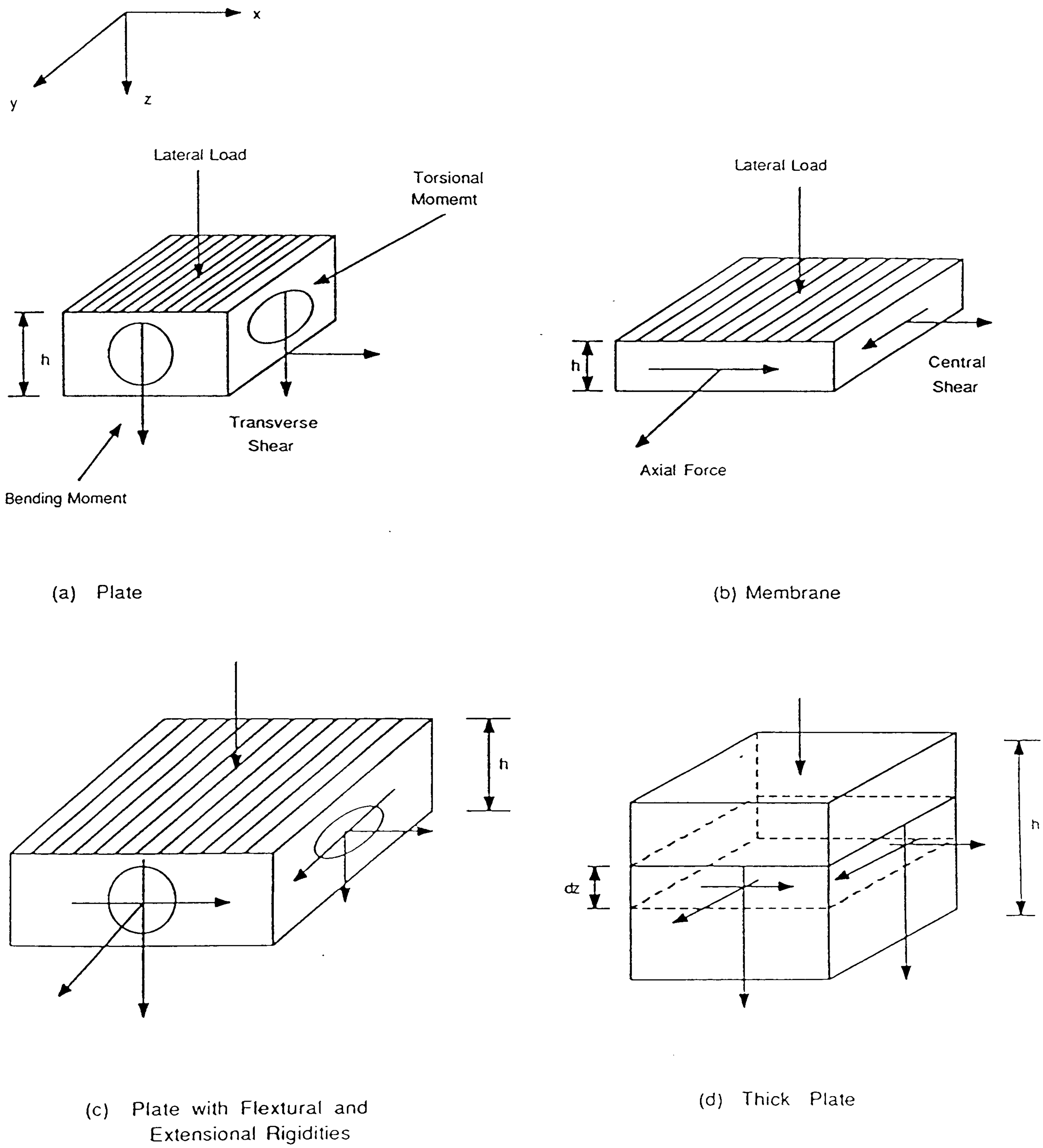
$$\frac{\partial \chi}{\partial \underline{\delta}} = \underline{K}_{eq}\ \underline{\delta} - \underline{F}_e = \underline{0} \tag{3.101}$$

i.e.

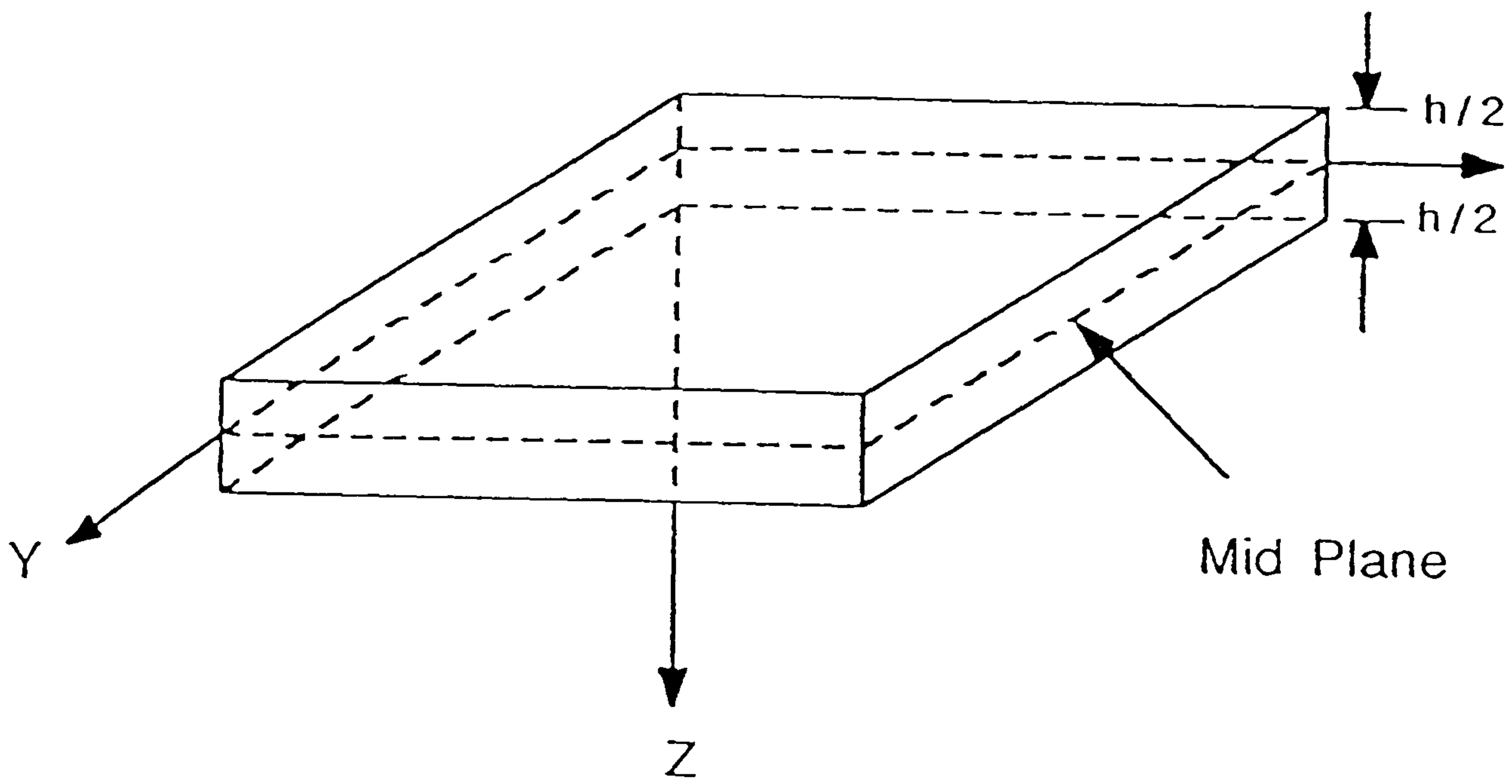
$$\underline{K}_{eq}\,\underline{\delta} = \underline{F}_e$$

and it can be proved from equations (3.100) and (3.101) that

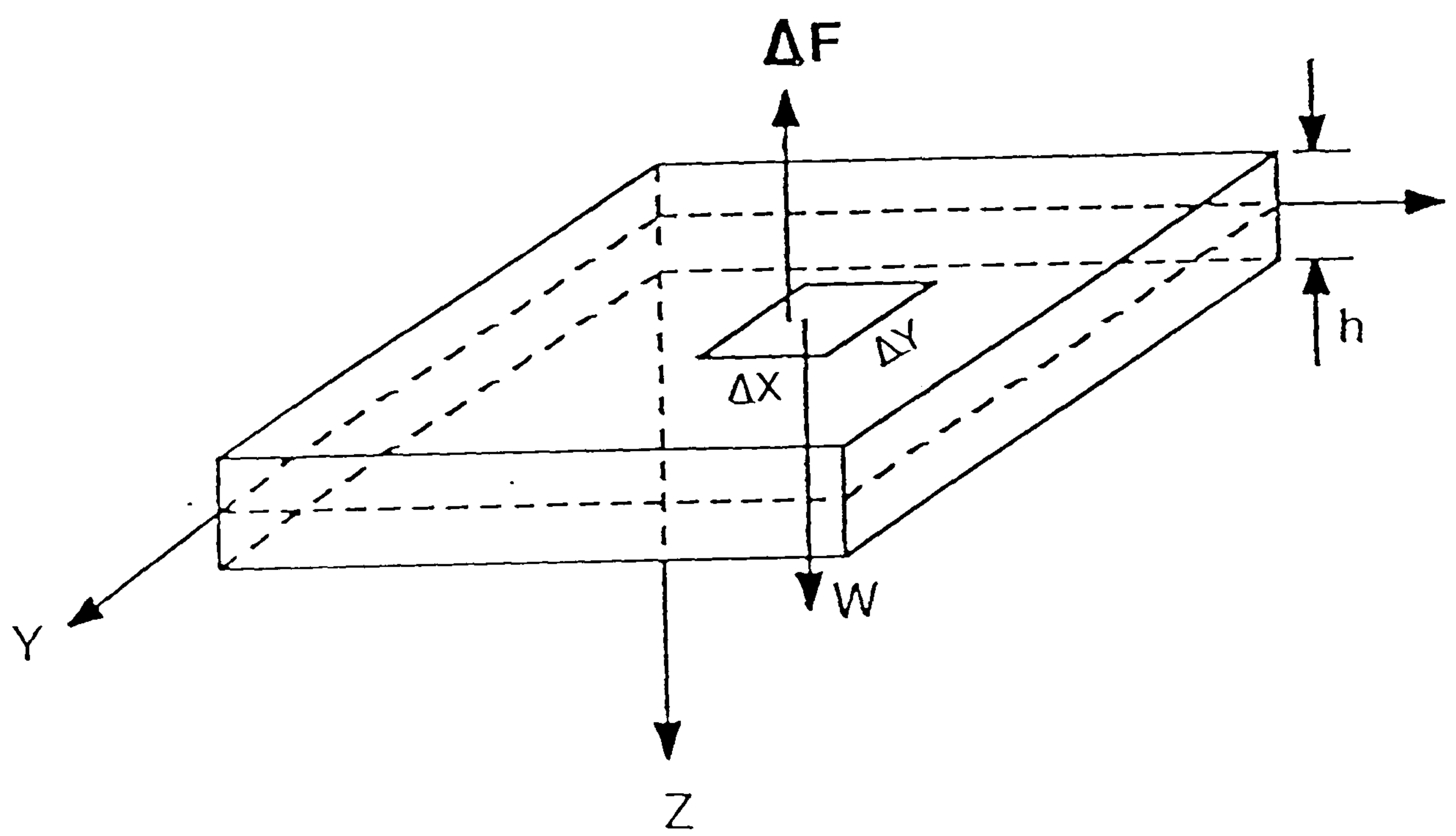
$$\underline{K}_{eq} = \begin{bmatrix} \underline{K}_o & \underline{0} & \underline{0} \\ \underline{0} & \underline{K}_b + \underline{K}_f & -\mathrm{K}_{bt} \\ \underline{0} & -\underline{K}_{bt}^t & \underline{K}_s + \underline{K}_t \end{bmatrix}.$$



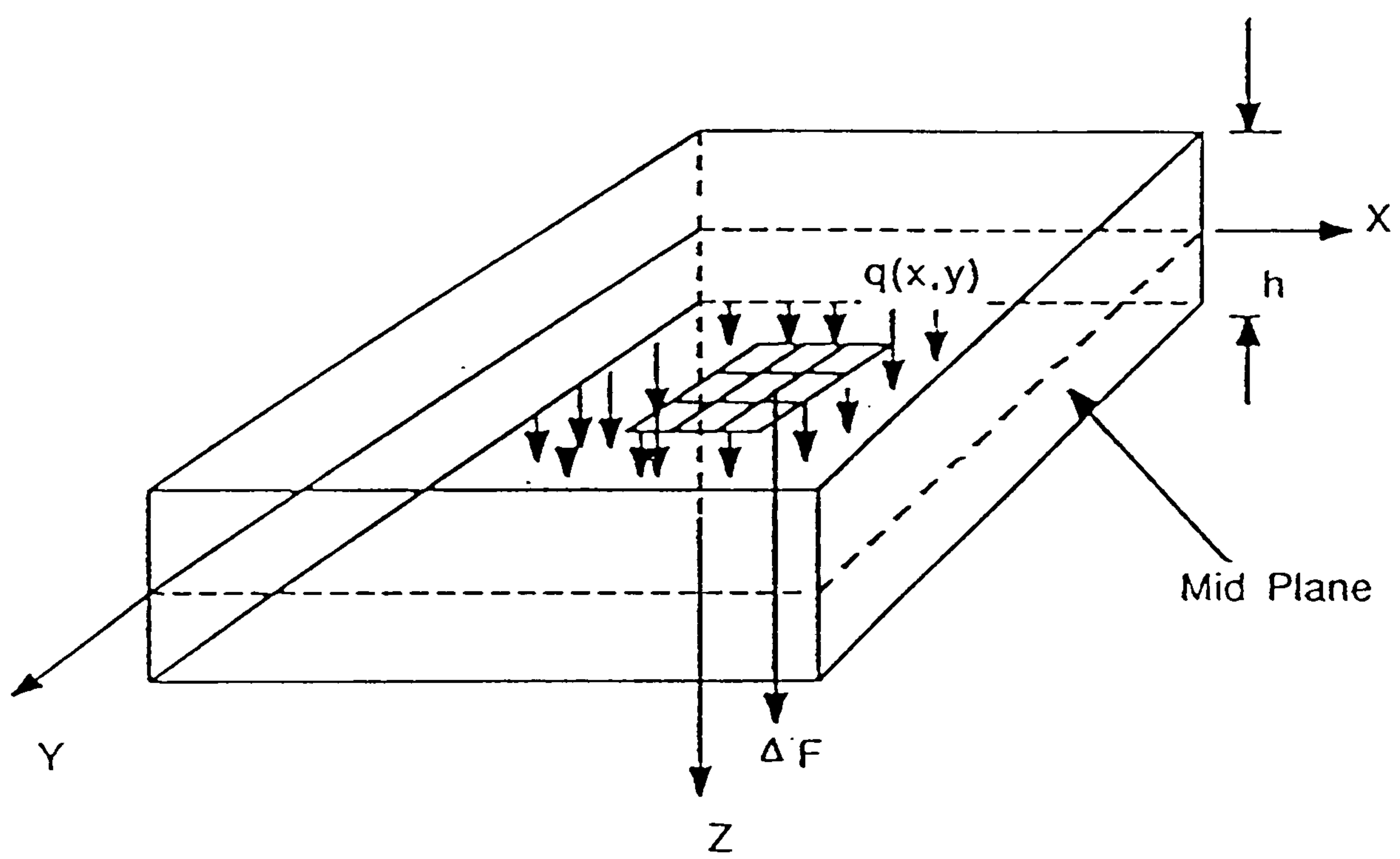
(Fig 3.1) Internal Forces in Various Types of Plate Bending Element



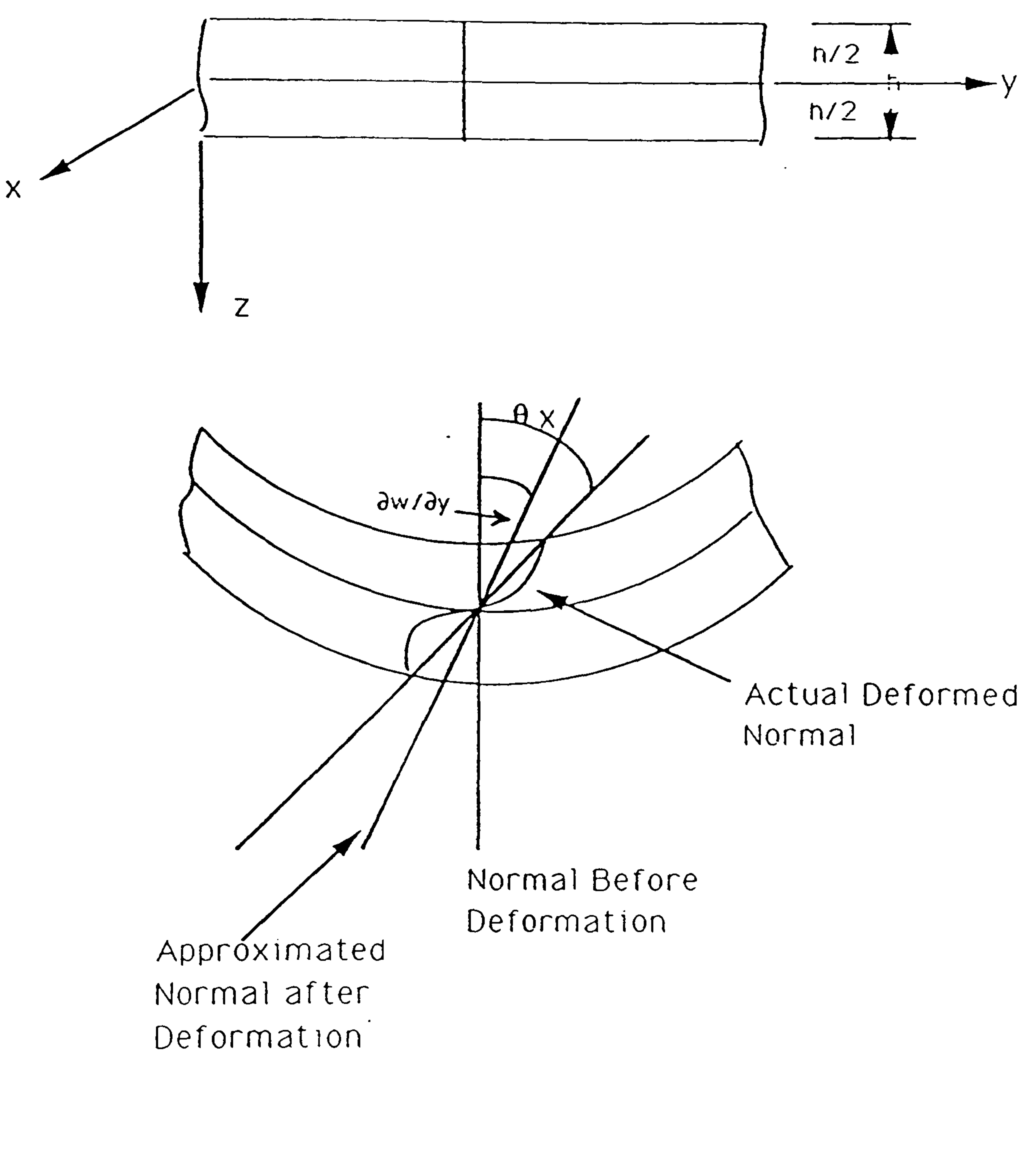
(Fig 3.2) Plate Bending Element



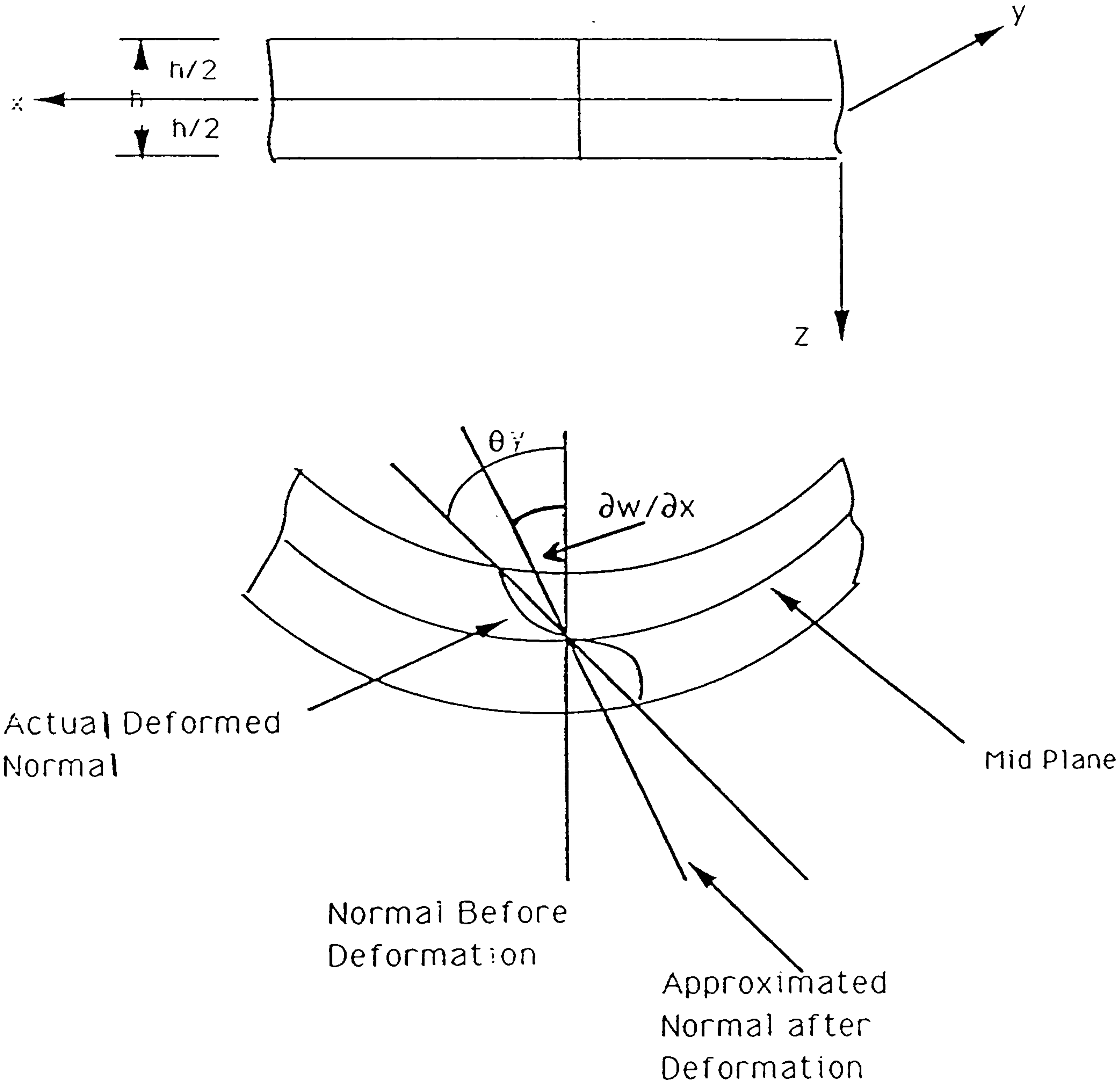
(Fig 3.3) An infinitesimal of Plate Bending Element



(Fig 3.4) Plate Subjected to Pressure Loading



(Fig 3.5.a) Plate Bending Displacement for Mindlin Element



(Fig. 3.5 b) Plate Bending Displacement for Mindlin Element

CHAPTER FOUR

BOUNDARY ELEMENT ANALYSIS OF THIN PLATES ON ELASTIC FOUNDATION

4.1 INTRODUCTION

This chapter introduces several new ideas for dealing with thin plates on an elastic Winkler Foundation. The first idea attempted here is the derivation of boundary integral equations, based upon three degrees-of-freedom per node (w, $\partial w/\partial x$, $\partial w/\partial y$). Such equations will not contain Kirchhoff corner forces and are, therefore, easy to program and easy to use for the evaluation of stresses and strains. An interesting new method for the derivation of fundamental solutions based upon a strain-function approach together with Fourier and Hankel integral transforms is introduced. It was found that some of the fundamental solution kernels of published derivations would lead to divergent integrals, and hence limiting the accuracy of the analysis.

A completely new idea for dealing with fundamental solutions of thin plates on an elastic Winkler foundation has been attempted and is introduced in this chapter. It is based upon separating terms representing the elastic foundation effect from those representing the plate bending problems without foundation. Using such an approach, it becomes possible to express the foundation effect parameters in terms of new "Modified" non-singular Kelvin functions, and hence reducing singularity problems to those of plate bending ones solved earlier by other authors [Ref 90].

4.2 DERIVATION OF BASIC BOUNDARY INTEGRAL EQUATIONS

4.2.1 Review of governing equations

4.2.1.1 Definitions of basic parameters

Consider a flat plate which is defined in terms of a mid-plane (in the x-y plane) and a thickness distribution $h(x,y)$, as shown in [Fig 4.1]. The upper surface of the plate is defined by $z=-\frac{h}{2}$, and it is assumed to be subjected to a pressure loading $q(x,y)$ acting in the z-direction, whilst the lower surface ($z=+\frac{h}{2}$) is resting on an elastic Winkler foundation with stiffness "K".

The stress analysis is based upon Kirchhoff's assumptions discussed in section 3.2. The parameters used in the governing equations are defined as follows:

a) Shear forces (per unit length):

The total Shear forces acting over the thickness per unit length in the y and x directions respectively, can be defined at any point (x,y) as follows:

$$Q_x(x,y) = \int_{-h/2}^{h/2} \tau_{xz}\, dz \tag{4.1}$$

$$Q_y(x,y) = \int_{-h/2}^{h/2} \tau_{yz}\, dz \tag{4.2}$$

b) Moments (per unit length):

To avoid difficulties with notation, the moments per unit length will be defined directly in terms of the corresponding stress components as follows:

$$M_x(x,y) = \int_{-h/2}^{h/2} z\, \sigma_x\, dz \tag{4.3}$$

$$M_y(x,y) = \int_{-h/2}^{h/2} z\, \sigma_y\, dz \tag{4.4}$$

$$M_{xy} = M_{yx} = \int_{-h/2}^{h/2} z\, \tau_{xy}\, dz \tag{4.5}$$

c) Surface moments and shear forces (per unit length):

Surface traction defined with respect to a normal to the outer boundary and z-axis; i.e with directional cosines (l, m, 0) as shown in Figure 4.1, are as follows:

$$\begin{aligned} T_x &= l\, \sigma_x + m\, \tau_{xy} \\ T_y &= l\, \tau_{xy} + m\, \sigma_y \\ T_z &= l\, \tau_{xz} + m\, \tau_{yz} \end{aligned}$$

Hence the following surface moments and shear force (per unit length) can be defined:

$$t_x = \int_{-h/2}^{h/2} z\, T_x\, dz = l\, M_x + m\, M_{xy} \tag{4.6}$$

$$t_y = \int_{-h/2}^{h/2} z\, T_y\, dz = l\, M_{xy} + m\, M_y \tag{4.7}$$

$$t_z = \int_{-h/2}^{h/2} T_z\, dz = l\, Q_x + m\, Q_y \tag{4.8}$$

4.2.1.2 Boundary conditions at upper and lower surfaces.

Considering the equilibrium conditions over the lower and upper surfaces of the plate ($z=\pm\frac{h}{2}$), it can be shown that:

$$\tau_{xz} = \tau_{yz} = 0 \tag{4.9a}$$

$$\sigma_z = -q \tag{4.9b}$$

$$\text{at } z = -\tfrac{h}{2}$$

similarly, it can be shown that:

$$\tau_{xz} = \tau_{yz} = 0 \tag{4.10a}$$

$$\sigma_z = -K\, w \tag{4.10b}$$

$$\text{at } z = \tfrac{h}{2}$$

where "w" is the deflection of the plate in the z-direction and "K" is the modulus of foundation elasticity.

4.2.1.3 Equilibrium equations over the plate thickness

The equations of equilibrium at any point (x,y,z) inside the plate are given as follows:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \tag{4.11a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \tag{4.11b}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \tag{4.11c}$$

where X, Y, Z are domain loading intensities (forces per unit volume). Using direct integration, it can be proved that:

$$\int_{-h/2}^{h/2} \frac{\partial \sigma_z}{\partial z}\, dz = (\sigma_z)_{h/2} - (\sigma_z)_{-h/2}$$

$$= -K\, w + q \tag{4.12}$$

and from integration by parts, it can be shown that:

$$\int_{-h/2}^{h/2} z\, \frac{\partial \tau_{xz}}{\partial z}\, dz = \Big(z\tau_{xz} \Big)_{-h/2}^{h/2} - \int_{-h/2}^{h/2} \tau_{xz}\, dz = -Q_x \tag{4.13}$$

and similarly,

$$\int_{-h/2}^{h/2} z\, \frac{\partial \tau_{yz}}{\partial z}\, dz = -Q_y \tag{4.14}$$

In the absence of domain loading (i.e X=Y=Z=0), equations (4.11) can be modified and integrated over the thickness as follows:

$$\int_{-h/2}^{h/2} z \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) dz = 0 \tag{4.15a}$$

$$\int_{-h/2}^{h/2} z \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) dz = 0 \tag{4.15b}$$

$$\int_{-h/2}^{h/2} \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} \right) dz = 0 \tag{4.15c}$$

Using the parameters defined by equations (4.1) to (4.5), and the results given by equations (4.12) to (4.14), the integrated equations (4.15) can be reduced as follows:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \tag{4.16a}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{4.16b}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q - Kw = 0 \tag{4.16c}$$

4.2.1.4 Transverse stresses:

From the boundary conditions given by equations (4.9) and (4.10), it is common practice in Kirchhoff's plate bending theory to ignore transverse stresses.

In the derivation presented here, although non-zero values for Q_x and Q_y may be considered in the analysis, the effect of transverse shear stresses on the deformation of the plate will be neglected. However, due to the pressure of the foundation, the σ_z effect will be considered.

From the analysis given in section 3.3.1, equation (3.19) and (3.20), transverse shear stresses can be assumed at any point (x,y,z) in the plate as follows:

$$\tau_{xz} = \Phi(x,y)\left(1 - \frac{4z^2}{h^2}\right) \tag{3.19}$$

$$\tau_{yz} = \Psi(x,y)\left(1 - \frac{4z^2}{h^2}\right) \tag{3.20}$$

using equations (4.1) and (4.2), it can be shown that:

$$Q_x = \int_{-h/2}^{h/2} \Phi(x,y)\left(1 - \frac{4z^2}{h^2}\right) dz = \frac{2h}{3}\Phi(x,y)$$

$$Q_y = \int_{-h/2}^{h/2} \Psi(x,y)\left(1 - \frac{4z^2}{h^2}\right) dz = \frac{2h}{3}\Psi(x,y)$$

i.e

$$\Phi(x,y) = \frac{3Q_x}{2h} \tag{4.17a}$$

and

$$\Psi(x,y) = \frac{3Q_y}{2h} \tag{4.17b}$$

Hence, the transverse shear-stress distributions over the thickness, can be expressed as follows:

$$\tau_{xz} = \frac{3Q_x}{2h}\left(1 - \frac{4z^2}{h^2}\right) \tag{4.18a}$$

$$\tau_{yz} = \frac{3Q_y}{2h}\left(1 - \frac{4z^2}{h^2}\right) \tag{4.18b}$$

Substituting from equations (4.18) into the following equilibrium equation:

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0,$$

it can be deduced that

$$\frac{\partial \sigma_z}{\partial z} = \frac{3}{2h}\left[(q - K w)(1 - \frac{4 z^2}{h^2}) \right] \tag{4.19}$$

Integrating equation (4.19) from z=0 to any z, and using the given boundary conditions, it can be shown that:

$$\sigma_z(x,y,z) = - \frac{(q + K w)}{2} + \frac{(q - K w)}{2 h}(3 z - \frac{4 z^3}{h^2}) \tag{4.20}$$

4.2.1.5 <u>Displacement, Strain and Stress Components.</u>

Using an approach similar to that used in section 3.2.1, the displacement components at any point (x,y,z) due to an out of plane loading can be expressed as follows:

$$u(x,y,z) \simeq - z \frac{\partial w}{\partial x} \tag{4.21a}$$

$$v(x,y,z) \simeq - z \frac{\partial w}{\partial y} \tag{4.21b}$$

$$w(x,y,z) \simeq w(x,y) \tag{4.21a}$$

From which the relevant strain components can be defined in terms of deflection "w", as follows:

$$\epsilon_x = \frac{\partial u}{\partial x} = - z \frac{\partial^2 w}{\partial x^2} \tag{4.22a}$$

$$\epsilon_y = \frac{\partial v}{\partial y} = - z \frac{\partial^2 w}{\partial y^2} \tag{4.22b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = - 2 z \frac{\partial^2 w}{\partial x \partial y} \tag{4.22c}$$

Using generalised Hooke's law, the strain components may be expressed in terms of stress components for linear elastic isotropic materials as

$$\epsilon_x = \frac{1}{E}\,(\sigma_x - \nu\sigma_y - \nu\sigma_z)$$

$$\epsilon_y = \frac{1}{E}\,(\sigma_y - \nu\sigma_x - \nu\sigma_z)$$

$$\tau_{xy} = \frac{2\,(\,1 + \nu\,)}{E}\,\gamma_{xy}$$

from which it can be deduced that:

$$\sigma_x = \frac{E}{(\,1 - \nu^2\,)}(\,\epsilon_x + \nu\epsilon_y\,) + \frac{\nu}{(\,1 - \nu\,)}\,\sigma_z \tag{4.23a}$$

$$\sigma_y = \frac{E}{(\,1 - \nu^2\,)}(\,\epsilon_y + \nu\epsilon_x\,) + \frac{\nu}{(\,1 - \nu\,)}\,\sigma_z \tag{4.23b}$$

$$\tau_{xy} = \frac{(\,1 - \nu\,)}{2}\frac{E}{(\,1 - \nu^2\,)}\,\gamma_{xy} \tag{4.23c}$$

Substituting from equations (4.22), (4.20) into (4.23) the stress components may be expressed in terms of deflection "w", as follows:

$$\sigma_x = -\frac{Ez}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right) + \frac{\nu}{2(1-\nu)}\left[-(q + K\,w) + \frac{(q - K\,w)}{h}(3\,z - \frac{4z^3}{h^2})\right] \tag{4.24a}$$

$$\sigma_y = -\frac{Ez}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}\right) + \frac{\nu}{2(1-\nu)}\left[-(q + K\,w) + \frac{(q - K\,w)}{h}(3\,z - \frac{4\,z^3}{h^2})\right] \tag{4.24b}$$

$$\tau_{xy} = -\,(\,1 - \nu)\,\frac{E\,z}{(1-\nu^2)}\,\frac{\partial^2 w}{\partial x \partial y} \tag{4.24c}$$

4.2.1.6 Moments and shears in terms of deflection "w"

Substituting from equations (4.24) into (4.3), (4.4), and (4.5), it can be shown that:

$$M_x = -\,D\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right) + \frac{\nu}{(1-\nu)\lambda^2}(q - K\,w) \tag{4.25a}$$

$$M_y = - D \left(\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{\nu}{(1-\nu)\lambda^2}(q - K\,w) \tag{4.25b}$$

$$M_{xy} = - (1-\nu)\, D \frac{\partial^2 w}{\partial x \partial y} \tag{4.25c}$$

where

$$D = \frac{E\,h^3}{12\,(1-\nu^2)}; \qquad \lambda^2 = \frac{10}{h^2}$$

Substituting from equations (4.25) into (4.16), the shear forces (per unit length) may be expressed as follows:

$$Q_x = - D \frac{\partial}{\partial x}(\nabla^2 w) + \frac{\nu}{(1-\nu)\lambda^2} \frac{\partial}{\partial x}(q - Kw) \tag{4.26a}$$

$$Q_y = - D \frac{\partial}{\partial y}(\nabla^2 w) + \frac{\nu}{(1-\nu)\lambda^2} \frac{\partial}{\partial y}(q - Kw) \tag{4.26b}$$

for q=0, or q=constant, the previous equations may be simplified as follows:

$$Q_x = - D \frac{\partial}{\partial x}(\nabla^2 + a^2)\, w \tag{4.27a}$$

$$Q_y = - D \frac{\partial}{\partial y}(\nabla^2 + a^2)\, w \tag{4.27b}$$

where

$$a^2 = \frac{\nu}{(1-\nu)\,\lambda^2} \frac{K}{D}$$

Substituting from equations (4.26) into (4.16c), a governing differential equation can be expressed in terms of "w" as follows:

$$(\, \nabla^4 + a^2\, \nabla^2 + \kappa^4 \,)\, w = \frac{1}{D}\left[q + \frac{\nu}{(1-\nu)\lambda^2}\, \nabla^2\, q \right] \tag{4.28}$$

which may be reduced for constant or linear "q" as follows:

$$(\, \nabla^4 + a^2\, \nabla^2 + \kappa^4 \,)\, w = \frac{q}{D} \tag{4.29}$$

where

$$\kappa^4 = \frac{K}{D}.$$

4.2.2 Weighted - residual expressions

Considering an approximate solution which satisfies the given boundary conditions, then a weighted-residual expression can be deduced from equilibrium equations (4.16) as follows:

$$\iint_\Omega \Big\{ \theta_x^* \Big(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x \Big) + \theta_y^* \Big(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y \Big)$$
$$+ w^* \Big(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q - K\, w\Big) \Big\}\, dx\, dy = 0 \tag{4.30}$$

where $u^*(x,y)$, $v^*(x,y)$, $w^*(x,y)$ are weighted functions, and "Ω" represents the x-y domain of the mid-plane of the plate which has boundary "Γ". Using the following integration by parts theorems:

$$\iint_\Omega f \frac{\partial g}{\partial x}\, dx\, dy = \oint_\Gamma f\, g\, l\, ds - \iint_\Omega \frac{\partial f}{\partial x}\, g\, dx\, dy \tag{4.31a}$$

$$\iint_\Omega f \frac{\partial g}{\partial y}\, dx\, dy = \oint_\Gamma f\, g\, m\, ds - \iint_\Omega \frac{\partial f}{\partial y}\, g\, dx\, dy \tag{4.31b}$$

where l and m represent the directional cosines of the normal to "Γ" (and the z-axis), in the outward direction with respect to "Ω", then equation (4.30) may be rewritten as follows:

$$\oint_\Gamma \Big(\theta_x^*\, t_x + \theta_y^*\, t_y + w^* t_z\Big)\, ds \;+\; \iint_\Omega w^*(q-Kw)\, dx\, dy$$

$$-\iint_\Omega \Big\{M_x \frac{\partial \theta_x^*}{\partial x} + M_{xy}\Big(\frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x}\Big) + M_y \frac{\partial \theta_y^*}{\partial y}\Big\}\, dx\, dy$$

$$-\iint_\Omega \Big\{Q_x\Big(\theta_x^* + \frac{\partial w^*}{\partial x}\Big) + Q_y\Big(\theta_y^* + \frac{\partial w^*}{\partial y}\Big)\Big\}\, dx\, dy = 0 \tag{4.32}$$

For thin plates the effects of transverse shears on deformation is negligible, and θ_x^* and θ_y^* may be selected so as to cancel Q_x and Q_y in the previous expression as follows:

$$\theta_x^* = -\frac{\partial w^*}{\partial x}; \qquad \theta_y^* = -\frac{\partial w^*}{\partial y}$$

and equation (4.32) may be reduced to

$$\oint_\Gamma \Big(\theta_x^* t_x + \theta_y^* t_y + w^* t_z\Big)\, ds \; + \iint_\Omega w^*(q-Kw)\, dx\, dy$$

$$+ \iint_\Omega \Big\{ M_x \frac{\partial^2 w^*}{\partial x^2} + 2M_{xy}\frac{\partial^2 w^*}{\partial x \partial y} + M_y \frac{\partial^2 w^*}{\partial y^2}\Big\}\, dx\, dy = 0 \tag{4.33}$$

Using equations (4.25), it can be proved that:

$$M_x \frac{\partial^2 w^*}{\partial x^2} + 2\, M_{xy}\frac{\partial^2 w^*}{\partial x \partial y} + M_y \frac{\partial^2 w^*}{\partial y^2}$$

$$= M_x^* \frac{\partial^2 w^*}{\partial x^2} + 2\, M_{xy}^* \frac{\partial^2 w^*}{\partial x \partial y} + M_y^* \frac{\partial^2 w^*}{\partial y^2} + \frac{\nu}{(1-\nu)\lambda^2}(q-Kw)\nabla^2 w^* \tag{4.34}$$

where M_x^*, M_y^* and M_{xy}^* are defined as follows:

$$M_x^* = -D\Big(\frac{\partial^2 w^*}{\partial x^2} + \nu\frac{\partial^2 w^*}{\partial y^2}\Big) \tag{4.35a}$$

$$M_y^* = -D\Big(\nu\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2}\Big) \tag{4.35b}$$

$$M_{xy}^* = -D(1-\nu)\frac{\partial^2 w^*}{\partial x \partial y} \tag{4.35c}$$

Hence it can be shown using equations (4.31) that:

$$\iint_\Omega \Big[M_x \frac{\partial^2 w^*}{\partial x^2} + 2\, M_{xy}\frac{\partial^2 w^*}{\partial x \partial y} + M_y \frac{\partial^2 w^*}{\partial y^2}\Big]\, dx\, dy$$

$$= -\oint_\Gamma \Big(\theta_x t_x^* + \theta_y t_y^*\Big)\, ds + \iint_\Omega \frac{\nu}{(1-\nu)\lambda^2}(q-Kw)\nabla^2 w^*\, dx\, dy$$

$$+ \iint_\Omega \Big\{ \theta_x\Big(\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y}\Big) + \theta_y\Big(\frac{\partial M_{xy}^*}{\partial x} + \frac{\partial M_y^*}{\partial y}\Big)\Big\}\, dx\, dy \tag{4.36}$$

where

$$t_x^* = l\, M_x^* + m\, M_{xy}^* \tag{4.37a}$$

$$t_y^* = l\,M_{xy}^* + m\,M_y^* \tag{4.37b}$$

$$\theta_x = -\frac{\partial w}{\partial x} \tag{4.37c}$$

$$\theta_y = -\frac{\partial w}{\partial y} \tag{4.37d}$$

It can also be shown that:

$$\iint_\Omega \left\{ M_x \frac{\partial^2 w^*}{\partial x^2} + 2M_{xy}\frac{\partial^2 w^*}{\partial x \partial y} + M_y \frac{\partial^2 w^*}{\partial y^2} \right\} dx\ dy$$

$$= -\oint_\Gamma \left(\theta_x t_x^* + \theta_y t_y^*\right) ds + \iint_\Omega \frac{\nu}{(1-\nu)\lambda^2}(q-Kw)\nabla^2 w^*\ dx\ dy$$

$$+ \iint_\Omega \left\{ \theta_x\left(\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - Q_x^*\right) + \theta_y\left(\frac{\partial M_{xy}^*}{\partial x} + \frac{\partial M_y^*}{\partial y} - Q_y^*\right)\right\} dx\ dy$$

$$+ \iint_\Omega \left(\theta_x Q_x^* + \theta_y Q_y^*\right) dx\ dy \tag{4.38}$$

where Q_x^* and Q_y^* are two additional unknown parameters inserted so as to have compatible inverse expressions.

Using integration by parts theorems equation (4.31), it can be shown that:

$$\iint_\Omega \left(\theta_x Q_x^* + \theta_y Q_y^*\right) dx\ \ dy = -\iint_\Omega \left(\frac{\partial w}{\partial x} Q_x^* + \frac{\partial w}{\partial y} Q_y^*\right) dx\ dy$$

$$= -\oint_\Gamma w\ t_z^* + \iint_\Omega w\left(\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y}\right) dx\ dy \tag{4.39}$$

where

$$t_z^* = l\,Q_x^* + m\,Q_y^* \tag{4.40}$$

Using equation (4.39) then equation (4.36) may be modified as follows:

$$\iint_{\Omega}\left\{M_x\frac{\partial^2 w^*}{\partial x^2} + 2M_{xy}\frac{\partial^2 w^*}{\partial x \partial y} + M_y\frac{\partial^2 w^*}{\partial y^2}\right\} dx\, dy$$

$$= -\oint_{\Gamma}\left(\theta_x t_x^* + \theta_y t_y^* + w t_z^*\right) ds + \iint_{\Omega}\frac{\nu}{(1-\nu)\lambda^2}(q-Kw)\nabla^2 w^*\, dx\, dy$$

$$+ \iint_{\Omega}\left\{\theta_x\left(\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - Q_x^*\right) + \theta_y\left(\frac{\partial M_{xy}^*}{\partial x} + \frac{\partial M_y^*}{\partial y} - Q_y^*\right) + w\left(\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y}\right)\right\} dx\, dy \quad (4.41)$$

Substituting from equation (4.41) into equation (4.33) then an inverse weighted residual expression may be obtained as follows:

$$\oint_{\Gamma}\left(\theta_x^* t_x + \theta_y^* t_y + w^* t_z\right) ds - \oint_{\Gamma}\left(\theta_x t_x^* + \theta_y t_y^* + w t_z^*\right) ds$$

$$+ \iint_{\Omega} q\left[w^* + \frac{\nu}{(1-\nu)\lambda^2}\nabla^2 w^*\right] dx\, dy$$

$$+ \iint_{\Omega}\left\{\theta_x\left(\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - Q_x^*\right) + \theta_y\left(\frac{\partial M_{xy}^*}{\partial x} + \frac{\partial M_y^*}{\partial y} - Q_y^*\right)\right.$$

$$\left. + w\left[\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - K\left(w^* + \frac{\nu}{(1-\nu)\lambda^2}\nabla^2 w^*\right)\right]\right\} dx\, dy = 0 \quad (4.42)$$

notice that

$$\frac{\nu}{(1-\nu)\lambda^2}\nabla^2 w^* = \frac{\nu h^2}{10(1-\nu)}\nabla^2 w^*$$

which is very small compared with w^* for the case of a very thin plate. Hence, the following approximation will be carried out in equation (4.42)

$$w^* + \frac{\nu}{(1-\nu)\lambda^2}\nabla^2 w^* \simeq w^* \quad (4.43)$$

and equation (4.42) will be simplified as follows

$$\oint_{\Gamma}\left(\theta_x^* \, t_x + \theta_y^* \, t_y + w^* \, t_z\right) ds - \oint_{\Gamma}\left(\theta_x t_x^* + \theta_y t_y^* + w t_z^*\right) ds$$

$$+ \iint_{\Omega} q w^* \, dx \, dy + \iint_{\Omega}\left\{\theta_x\left(\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - Q_x^*\right) + \theta_y\left(\frac{\partial M_{xy}^*}{\partial x} + \frac{\partial M_y^*}{\partial y} - Q_y^*\right)\right.$$

$$\left. + w\left(\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - K w^*\right)\right\} dx \, dy = 0 \qquad (4.44)$$

4.2.3 Derivation of fundamental solution:

4.2.3.1 Strain function method for fundamental solution.

A fundamental solution can be obtained in terms of Q_x^*, Q_y^* and w^* which are governed by the following equations:

$$\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - Q_x^* = - e_x \delta(x - x_i, y - y_i) \qquad (4.45a)$$

$$\frac{\partial M_{xy}^*}{\partial x} + \frac{\partial M_y^*}{\partial y} - Q_y^* = - e_y \delta(x - x_i, y - y_i) \qquad (4.45b)$$

$$\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - K w^* = - e_z \delta(x - x_i, y - y_i) \qquad (4.45c)$$

where

$$M_x^* = - D\left(\frac{\partial^2 w^*}{\partial x^2} + \nu \frac{\partial^2 w^*}{\partial y^2}\right)$$

$$M_y^* = - D\left(\nu \frac{\partial^2 w^*}{\partial y^2} + \frac{\partial^2 w^*}{\partial x^2}\right)$$

$$M_{xy}^* = - (1 - \nu) D \frac{\partial^2 w^*}{\partial x \partial y}$$

$\delta(x - x_i, y - y_i) \equiv \delta(x - x_i)\, \delta(y - y_i)$ is a two dimensional Dirac-Delta function,

e_x, e_y and e_z are arbitrary constants.

Hence, equation (4.45) may be rewritten in terms of Q_x^*, Q_y^* and w^* as follows:

$$\underset{\sim}{\mathfrak{D}}\begin{bmatrix} Q_x^* \\ Q_y^* \\ Dw^* \end{bmatrix} = \delta(x - x_i\ ,\ y - y_i)\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \tag{4.46}$$

where

$$\underset{\sim}{\mathfrak{D}} = \begin{bmatrix} 1 & 0 & \frac{\partial}{\partial x}\nabla^2 \\ 0 & 1 & \frac{\partial}{\partial y}\nabla^2 \\ -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & \kappa^4 \end{bmatrix} \tag{4.47}$$

and

$$\kappa^4 = \frac{K}{D}$$

To simplify the differential equations described by means of the matrix equation (4.46), three strain functions f_1, f_2 and f_3 will be assumed such that

$$\begin{bmatrix} Q_x^* \\ Q_y^* \\ Dw^* \end{bmatrix} = \underset{\sim}{\mathfrak{C}}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{4.48}$$

where $\underset{\sim}{\mathfrak{C}}$ is a differential operator matrix. Hence, equation (4.46) may be rewritten as follows:

$$\underset{\sim}{\mathfrak{D}}\ \underset{\sim}{\mathfrak{C}}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \delta(x - x_i\ ,\ y - y_i)\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \tag{4.49}$$

An approach similar to the "Galerkin vector approach" could be adopted for the definition of the $\underset{\sim}{\mathfrak{C}}$ matrix, by selecting it such that

$$\underset{\sim}{\mathfrak{D}}\ \underset{\sim}{\mathfrak{C}} = \begin{bmatrix} \mathcal{E}_1 & 0 & 0 \\ 0 & \mathcal{E}_2 & 0 \\ 0 & 0 & \mathcal{E}_3 \end{bmatrix} \quad . \tag{4.50}$$

i.e it leads to $\underset{\sim}{\mathfrak{D}}\ \underset{\sim}{\mathfrak{C}}$ being reduced to a diagonal matrix, and the matrix equation (4.49) may therefore, be decoupled in terms of the following three equations:

$$\mathcal{E}_1\, f_1 = e_x\, \delta(x - x_i, y - y_i) \tag{4.51a}$$
$$\mathcal{E}_2\, f_2 = e_y\, \delta(x - x_i, y - y_i) \tag{4.51b}$$
$$\mathcal{E}_3\, f_3 = e_z\, \delta(x - x_i, y - y_i) \tag{4.51c}$$

notice that from matrix properties, if $\underset{\sim}{A}^*$ is the cofactor matrix of a square matrix $\underset{\sim}{A}$, and $\underset{\sim}{A}$ is of n^{th} order, then

$$\underset{\sim}{A}(\underset{\sim}{A}^*)^t = (\underset{\sim}{A}^*)^t\underset{\sim}{A} = |\underset{\sim}{A}|\, \underset{=}{I}_{n\times n}$$

where $|\underset{\sim}{A}|$ is the determinant of matrix $\underset{\sim}{A}$ and $\underset{=}{I}_{n\times n}$ is the unit matrix of n^{th} order. Hence an optimum choice for the $\underset{\sim}{\mathfrak{C}}$ matrix is such that

$$\underset{\sim}{\mathfrak{C}} = (\underset{\sim}{\mathfrak{D}}^*)^t \tag{4.52}$$

which will lead to:

$$\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = |\underset{\sim}{\mathfrak{D}}| \tag{4.53}$$

and it is clear that the strain functions f_1, f_2 and f_3 can be expressed in terms of one function f, such that:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = f \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \tag{4.54}$$

and from equations (4.51) and (4.54), it can be proved that function f is governed by the following differential equation:

$$|\underset{\sim}{\mathfrak{D}}| f = \delta(x - x_i, y - y_i) \tag{4.55}$$

and it is clear from equation (4.54) and (4.48) that:

$$\begin{bmatrix} Q_x^* \\ Q_y^* \\ Dw^* \end{bmatrix} = \underset{\sim}{C} \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} f \qquad (4.56)$$

This approach, which is similar to the Hömmander method by [Ref 90], has reduced the governing differential equations of the fundamental solution parameters to one differential equation only (equation 4.55) and other fundamental solution parameters are obtained by differentiation.

From the definition of the $\underset{\sim}{\mathfrak{D}}$ matrix, (equation 4.47), it can be shown that:

$$\underset{\sim}{C} = (\underset{\sim}{\mathfrak{D}}^*)^t = \begin{bmatrix} \frac{\partial^2}{\partial y^2}\nabla^2 + \kappa^4 & -\frac{\partial^2}{\partial y \partial x}\nabla^2 & -\frac{\partial}{\partial x}\nabla^2 \\ -\frac{\partial^2}{\partial x \partial y}\nabla^2 & \frac{\partial^2}{\partial x^2}\nabla^2 + \kappa^4 & -\frac{\partial}{\partial y}\nabla^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 1 \end{bmatrix} \qquad (4.57)$$

and

$$|\underset{\sim}{\mathfrak{D}}| = \nabla^4 + \kappa^4 \qquad (4.58)$$

i.e equation (4.55) can be rewritten as follows

$$(\nabla^4 + \kappa^4)f = \delta(x - x_i,\, y - y_i) \qquad (4.59)$$

which will be referred to as the basic fundamental solution equation.

4.2.3.2 Solution of the basic fundamental solution equation.

A shifted two dimensional Fourier transform for a function $f(x - x_i, y - y_i)$ can be defined as follows: [Ref 135,140].

$$F(\xi,\eta) = \mathfrak{F}\left[f(x-x_i,y-y_i);\, \xi,\, \eta\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-x_i,y-y_i)e^{i[\xi(x - x_i) + \eta(y - y_i)]}dxdy \qquad (4.60)$$

and from the properties of the two dimensional Dirac delta function [Ref 133], it can be proved that:

$$\mathcal{F}\left[\delta(x - x_i, y - y_i);\xi,\eta\right] = \frac{1}{2\pi} \tag{4.61}$$

Notice that in the infinite domain equation (4.59) leads to $f(x - x_i,\ y - y_i)$ being $f(r)$ where

$$r = \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

Using polar coordinates, such that:

$$x = x_i + r\cos\theta; \qquad \xi = \zeta\cos\phi$$
$$y = y_i + r\sin\theta; \qquad \eta = \zeta\sin\phi$$

then equation (4.60) can be reduced to

$$\mathcal{F}(\xi,\eta) \equiv \mathcal{F}(\zeta) = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} f(r)\, e^{i\zeta r\cos(\theta - \phi)}\, r\, dr\, d\theta$$

$$\equiv \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} r\, f(r)\, e^{i\zeta r\cos\theta}\, dr\, d\theta \tag{4.62}$$

From [Ref 137], it can be proved that

$$\int_0^{2\pi} e^{i\zeta r\cos\theta}\, d\theta = 2\pi\, J_o(\zeta r) \tag{4.63}$$

and equation (4.62) can be reduced to:

$$\mathcal{F}(\zeta) = \bar{f}(\zeta) = \int_0^\infty r\, f(r)\, J_o(\zeta r)\, dr \tag{4.64}$$

where $\bar{f}(\zeta)$ is the Hankel's transform of f(r) as defined by equation (4.64). Due to the radial symmetry of the function f, it can be shown that:

$$\nabla^2 f(r) \equiv \frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr}$$

and from the properties of Hankel transforms [Ref 134,135], it can be shown that:

$$\int_0^\infty r[\nabla^2 f(r)]\, J_o(\zeta r)\, dr = -\zeta^2\bar{f}(\zeta) \tag{4.65}$$

and

$$\int_0^\infty r\left(\nabla^4 f(r)\right) J_o(\varsigma r)\ dr = \varsigma^4 \bar{f}(\varsigma) \tag{4.66}$$

Applying the two dimensional Fourier transform, as described above to equation (4.59) and using the results given by equations (4.61) and (4.66), it can be shown that:

$$(\varsigma^4 + \kappa^4)\ \bar{f}(\varsigma) = \frac{1}{2\pi}$$

i.e

$$\bar{f}(\varsigma) = \frac{1}{2\pi(\varsigma^4 + \kappa^4)}$$

or

$$\bar{f}(\varsigma) = \frac{1}{2\pi}\frac{1}{2i\kappa^2}\left(\frac{1}{\varsigma^2 + (-i)\kappa^2} - \frac{1}{\varsigma^2 + i\kappa^2}\right) \tag{4.67}$$

Notice that, the inverse transform of $\bar{f}(\varsigma)$ can be defined as follows [Ref 133]

$$f(r) = \int_0^\infty \varsigma\ \bar{f}(\varsigma)\ J_o(\varsigma r)\ d\varsigma \tag{4.68}$$

and from [Ref 133]

$$\int_0^\infty \frac{\varsigma}{\varsigma^2 + a^2} J_o(\varsigma r)\ d\varsigma = K_o(a r) \tag{4.69}$$

From equation (4.69), the inverse transform of equation (4.67), can be expressed as follows:

$$f(r) = \frac{1}{2\pi}\frac{1}{2i\kappa^2}\left[K_o\left(\sqrt{-i}\ \kappa r\right) - K_o\left(\sqrt{i}\ \kappa r\right)\right] \tag{4.70}$$

From the properties of Kelvin functions [Ref 137]

$$K_{er}(z) + iK_{ei}(z) = K_o(z\sqrt{i})$$

$$K_{er}(z) - iK_{ei}(z) = K_o(z\sqrt{-i})$$

i.e.

$$K_o(z\sqrt{-i}) - K_o(z\sqrt{i}) = -2iK_{ei}(z) \tag{4.71}$$

Substituting from equation (4.71) into (4.70), it can be proved that:

$$f(r) = -\frac{1}{2\pi\kappa^2}K_{ei}(z) \tag{4.72}$$

where

$$z = \kappa r$$

4.2.4 Boundary integral equations

Using equation (4.45), the inverse weighted residual expression (equation 4.44) can be simplified as follows:

$$c_i\Big((\theta_x)_i e_x + (\theta_y)_i e_y + w_i e_z\Big) + \oint_\Gamma \Big(t_x^* \theta_x + t_y^* \theta_y + t_z^* w\Big)\, ds$$

$$= \oint_\Gamma \Big(\theta_x^* t_x + \theta_y^* t_y + w^* t_z\Big)\, ds + \iint_\Omega w^* q\, dx\, dy \tag{4.73}$$

where

c_i	$= 0$	if (x_i, y_i) is outside Ω
	$= 0.5$	if (x_i, y_i) is on Γ
	$= 1.0$	if (x_i, y_i) is inside Ω

To facilitate the application of boundary conditions, slopes with respect to x and y axis can be expressed in terms of slopes with respect to normal and tangential directions as follows [Ref 90]:

$$t_x^* \theta_x + t_y^* \theta_y = -\Big(M_n^* \frac{\partial w}{\partial n} + M_{tn}^* \frac{\partial w}{\partial t}\Big) \tag{4.74a}$$

$$t_x \theta_x^* + t_y \theta_y^* = -\Big(M_n \frac{\partial w^*}{\partial n} + M_{tn} \frac{\partial w^*}{\partial t}\Big) \tag{4.74b}$$

where

$$w^* = (e_x\frac{\partial}{\partial x} + e_y\frac{\partial}{\partial y} + e_z)\frac{f}{D} \tag{4.75a}$$

$$\frac{\partial w^*}{\partial n} = l\,\frac{\partial w^*}{\partial x} + m\,\frac{\partial w^*}{\partial y} \tag{4.75b}$$

$$\frac{\partial w^*}{\partial t} = -m\,\frac{\partial w^*}{\partial x} + l\,\frac{\partial w^*}{\partial y} \tag{4.75c}$$

and

$$M_n^* = -D\left(\frac{\partial^2 w^*}{\partial n^2} + \nu\,\frac{\partial^2 w^*}{\partial t^2}\right) \tag{4.76a}$$

$$M_{tn}^* = -D\,(1-\nu)\,\frac{\partial^2 w^*}{\partial n \partial t} \tag{4.76b}$$

$$t_z^* = Q_n^* = -D\frac{\partial}{\partial n}(\nabla^2 w^*) \tag{4.76c}$$

hence, equation (4.73) may be modified as follows:

$$c_i\Big((\theta_x)_i e_x + (\theta_y)_i e_y + w_i e_z\Big) + \oint_\Gamma\left(Q_n^*\,w - M_n^*\,\frac{\partial w}{\partial n} - M_{tn}^*\,\frac{\partial w}{\partial t}\right) ds$$

$$= \oint_\Gamma\left(Q_n\,w^* - M_n\,\frac{\partial w^*}{\partial n} - M_{tn}\,\frac{\partial w^*}{\partial t}\right) ds + \iint_\Omega w^* q\,dx\,dy \tag{4.77}$$

Fundamental solution parameters may be defined such that:

$$\begin{bmatrix} w^* \\ \frac{\partial w^*}{\partial n} \\ \frac{\partial w^*}{\partial t} \end{bmatrix} = \underset{\sim}{U} \begin{bmatrix} e_z \\ e_x \\ e_y \end{bmatrix} \tag{4.78}$$

$$\begin{bmatrix} Q_n^* \\ M_n^* \\ M_{tn}^* \end{bmatrix} = \underset{\sim}{T} \begin{bmatrix} e_z \\ e_x \\ e_y \end{bmatrix} \tag{4.79}$$

Hence, it can be deduced from equations (4.75), (4.76), (4.78) and (4.79) that:

$$\underset{\sim}{U} = \frac{1}{D}\begin{bmatrix} f & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial n} & \frac{\partial^2 f}{\partial n \partial x} & \frac{\partial^2 f}{\partial n \partial y} \\ \frac{\partial f}{\partial t} & \frac{\partial^2 f}{\partial t \partial x} & \frac{\partial^2 f}{\partial t \partial y} \end{bmatrix} \tag{4.80}$$

and

$$\underset{\sim}{T} = \begin{bmatrix} F_1 & \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ F_2 & \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \\ F_3 & \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} \end{bmatrix} \tag{4.81}$$

where

$$F_1 = - \frac{\partial}{\partial n}(\nabla^2 f) \tag{4.82a}$$

$$F_2 = - (\frac{\partial^2 f}{\partial n^2} + \nu \frac{\partial^2 f}{\partial t^2}) \tag{4.82b}$$

$$F_3 = - (1-\nu) \frac{\partial^2 f}{\partial n \partial t} \tag{4.82c}$$

Explicit expressions for $\underset{\sim}{U}$ and $\underset{\sim}{T}$ matrices are listed in tables 4.1 and 4.2 respectively, and some useful details are given in Appendix C.

Substituting from equations (4.78), (4.79) into (4.77) and using the fact that e_x, e_y and e_z are arbitrary parameters, the following three boundary integral equations can be deduced with respect to the source point (x_i,y_i)

$$c_i w_i + \oint_\Gamma \left\{T_{11} W - T_{21}\left(\frac{\partial w}{\partial n}\right) - T_{31}\left(\frac{\partial w}{\partial t}\right)\right\} ds$$

$$= \oint_\Gamma \left\{U_{11} Q_n - U_{21} M_n - U_{31} M_{tn}\right\} ds + \iint_\Omega q U_{11} dx\, dy \tag{4.83a}$$

$$- c_i\left(\frac{\partial w}{\partial x}\right)_i + \oint_\Gamma\left\{T_{12}W - T_{22}\left(\frac{\partial w}{\partial n}\right) - T_{32}\left(\frac{\partial w}{\partial t}\right)\right\}ds$$

$$= \oint_\Gamma\left\{U_{12}Q_n - U_{22}M_n - U_{32}M_{tn}\right\}ds + \iint_\Omega qU_{12}dx\,dy \tag{4.83b}$$

$$- c_i\left(\frac{\partial w}{\partial y}\right)_i + \oint_\Gamma\left\{T_{13}W - T_{23}\left(\frac{\partial w}{\partial n}\right) - T_{33}\left(\frac{\partial w}{\partial t}\right)\right\}ds$$

$$= \oint_\Gamma\left\{U_{13}Q_n - U_{23}M_n - U_{33}M_{tn}\right\}ds + \iint_\Omega qU_{13}dx\,dy \tag{4.83c}$$

4.2.5 Domain loading terms

From previous analysis, it is clear that the domain loading in the boundary integral equations (equation (4.83)), may be defined as follows:

$$L_1 = \iint_\Omega qU_{11}dx\,dy = \iint_\Omega \frac{q}{D}f\,dx\,dy \tag{4.84a}$$

$$L_2 = \iint_\Omega qU_{12}dx\,dy = \iint_\Omega \frac{q}{D}\frac{\partial f}{\partial x}\,dx\,dy \tag{4.84b}$$

$$L_3 = \iint_\Omega qU_{13}dx\,dy = \iint_\Omega \frac{q}{D}\frac{\partial f}{\partial y}\,dx\,dy \tag{4.84c}$$

From the properties of Kelvin functions, [Ref 136], it can be shown that:

$$f = \nabla^2 Q \tag{4.85a}$$

where

$$Q = \frac{1}{2\pi\kappa^4}K_{er}(z) \tag{4.85b}$$

In order to obtain non - divergent integrals, L_2 and L_3 will be reduced as follows:

$$L_2 = \oint_\Gamma l\,q\,f\,ds - \iint_\Omega \frac{\partial q}{\partial x}\,f\,dx\,dy \tag{4.86a}$$

$$L_2 = \oint_\Gamma m\, q\, f\, ds \; - \iint_\Omega \frac{\partial q}{\partial y}\, f\, dx\, dy \tag{4.86b}$$

Substituting from equation (4.85) into (4.84a) and (4.86) and integrating the resulting expressions by parts, then the domain loading terms can be expressed in terms of boundary integral expressions as follows:

$$L_1(x_i,y_i) = \frac{1}{D}\left\{\oint_\Gamma q\, \frac{\partial Q}{\partial n}\, ds \; - \oint_\Gamma \frac{\partial q}{\partial n}\, Q\, ds\right\} \tag{4.87a}$$

$$L_2(x_i,y_i) = \frac{1}{D}\left\{\oint_\Gamma l\, q\, f\, ds \; - \oint_\Gamma \frac{\partial q}{\partial x}\, \frac{\partial Q}{\partial n}\, ds\right\} \tag{4.87b}$$

$$L_3(x_i,y_i) = \frac{1}{D}\left\{\oint_\Gamma m\, q\, f\, ds \; - \oint_\Gamma \frac{\partial q}{\partial y}\frac{\partial Q}{\partial n}\, ds\right\} \tag{4.87c}$$

where

$$\frac{\partial Q}{\partial n} = \frac{1}{2\pi\kappa^3}\, \frac{\partial}{\partial n}\, K'_{er}(z)$$

and for constant q:

$$\frac{\partial q}{\partial n} = \frac{\partial q}{\partial x} = \frac{\partial q}{\partial y} = 0$$

4.2.6 Boundary - element solution

Using constant boundary elements, as shown in Fig 4.2, integrals over the boundary can be described as follows:

$$\oint_\Gamma T_{11}\, w\, ds \; = \sum_{j=1}^{m}\Big(\int_{element\ j} T_{11}\, ds \Big) w_j$$

$$\oint_\Gamma U_{11}\, Q_n\, ds \; = \sum_{j=1}^{m}\Big(\int_{element\ j} U_{11}\, ds \Big)(Q_n)_j$$

where, m = Number of boundary elements and taking source point (x_i,y_i) to be mid - side nodes of all boundary elements, a system of linear algebraic simultaneous equations can be developed from integral equations (4.83) as follows:

$$(\underset{\sim}{C} + \underset{\sim}{H})\underset{\sim}{\delta} = \underset{\sim}{G}\underset{\sim}{t} + \underset{\sim}{l} \tag{4.88}$$

where

$$C_{3(i-1)+\alpha,3(j-1)+\beta} = C_i\delta_{ij}\ \delta_{\alpha\beta} \tag{4.89}$$

$$H_{3(i-1)+\alpha,3(j-1)+\beta} = \int_j T_{\beta,\alpha}(x - x_i, y - y_i)ds \tag{4.90}$$

$$G_{3(i-1)+\alpha,3(j-1)+\beta} = \int_j U_{\beta,\alpha}(x - x_i, y - y_i)ds \tag{4.91}$$

$$l_{3(i-1)+\alpha} = L_\alpha(x_i, y_i) \tag{4.92}$$

$$\underset{\sim}{\delta} = \left\{w_1\ w_{1,x}\ w_{1,y}\ \cdots\cdots\ w_m\ w_{m,x}\ w_{m,y}\right\}$$

$$\underset{\sim}{t} = \left\{(Q_n)_1, (M_n)_1, (M_{tn})_1, \cdots\cdots (Q_n)_m, (M_n)_m, (M_{tm})_m\right\}$$

and

$$i = 1, 2, \cdots m$$
$$j = 1, 2, \cdots m$$
$$\alpha = 1, 2, 3$$
$$\beta = 1, 2, 3$$

i.e

$$\hat{\underset{\sim}{H}}\ \underset{\sim}{\delta} = \underset{\sim}{G}\ \underset{\sim}{t} + \underset{\sim}{l} \tag{4.93}$$

where

$$\hat{H} = \underset{\sim}{C} + \underset{\sim}{H}$$

Matrix equation (4.93) represents 3m equations, and hence 3m of $\underset{\sim}{\delta}$ and/or $\underset{\sim}{t}$ should be specified as boundary conditions. The procedure for the application of boundary conditions and solution for unknown boundary values is similar to what is given in Ref [Ref 134].

Having determined all the boundary values, then displacement stresses and strains can be obtained at internal and boundary nodes, using procedures similar to those described in [Ref 89,90].

In equation (4.90), (4.91), when $i = j$, i.e when the source point is in the element over which the integrals are carried out some of the integrals become singular (i.e with infinite values within the region of integration). A full analysis of singular terms existing in such integral terms is given in Appendix D.

4.3 DERIVATION BASED UPON NEW "MODIFIED KELVIN FUNCTIONS"

4.3.1 Introduction

In the analysis of singular integral terms, of equations (4.90), (4.91), as described in Appendix D, it was noticed that the following expressions

$$\int_i T_{12}(x - x_i, y - y_i)\, ds = \frac{\kappa l}{\pi} \int_0^{z_o} \frac{K'_{er}(z)}{z} dz \tag{4.94}$$

$$\int_i T_{13}(x - x_i, y - y_i)\, ds = \frac{\kappa m}{\pi} \int_0^{z_o} \frac{K'_{er}(z)}{z} dz \tag{4.95}$$

have divergent integrals. Corresponding terms, with divergent integrals, occur in the BEM analysis of thin plates and one of the simple ways to get rid of them is to apply rigid translation and rotation conditions [Ref 89,90]

Unfortunately, such an approach is not possible to employ for plates on an elastic Winkler foundation, and the previous BEM formulation, given in section 4.2 is only applicable to plates with fixed or simply supported edge conditions, although some authors have never mentioned this limitation [Ref 115,120]. A new idea is introduced in this section, based on splitting the problem into two super-imposed problems; the plate bending problem without foundation and the foundation effect. The terms for each problem will be separated and the singular terms for each case will be analysed independently.

4.3.2 Derivation of modified fundamental solution

From the previous analysis, it is clear from equation (4.75) and (4.76) that the fundamental solution parameters used in the BEM solution are all functions of w^* which is defined in terms of a single function f, i.e (equation 4.75a)

$$w^* = \left(e_x\frac{\partial}{\partial x} + e_y\frac{\partial}{\partial y} + e_z\right)\frac{f}{D}$$

and f is governed by the following differential equation (equation 4.59)

$$(\nabla^4 + \kappa^4)f = \delta(x - x_i, y - y_i)$$

which has the following solution equation (4.72)

$$f(r) = -\frac{1}{2\pi\kappa^2}K_{ei}(z)$$

where

$$z = \kappa r$$

For the special case of no foundation, i.e $\kappa = 0$, equation (4.75) (b), (c) and (4.76) are still valid, but:

$$w^* = \left(e_x\frac{\partial}{\partial x} + e_y\frac{\partial}{\partial y} + e_z\right)\frac{f_1}{D} \tag{4.96}$$

where

$$\nabla^4 f_1 = \delta(x - x_i, y - y_i) \tag{4.97}$$

which has the following solution:

$$f_1 = \frac{1}{2\pi}\left[\ \frac{r^2}{4}(\log(r) - 1) + Cr^2\right] \tag{4.98}$$

where C is an arbitrary constant. With an appropriate selection of the parameter C, equation (4.98) may be rewritten as follows:

$$f_1 = \frac{1}{2\pi\kappa^2}\left\{\frac{z^2}{4}(\log(\frac{z}{2}) - 1)\right\} \tag{4.99}$$

Consider a function f_2 so as to represent the effect of the presence of the elastic foundation, such that:

$$f = f_1 + f_2 \tag{4.100}$$

then, it can be deduced from equations (4.72), (4.99) and (4.100) that:

$$f_2 = f - f_1 = \frac{1}{2\pi\kappa^2}\Big\{K_{ei}(z) + \frac{z^2}{4}(\log(\tfrac{z}{2}) - 1)\Big\} \tag{4.101}$$

Equation (4.101) has inspired us to define a new "Modified" Kelvin function as follows:

$$K_{eim}(z) = K_{ei}(z) + \frac{z^2}{4}(\log(\tfrac{z}{2}) - 1)\Big\} \tag{4.102}$$

i.e

$$f_2 = \frac{1}{2\pi\kappa^2}K_{eim}(z) \tag{4.103}$$

for a radially symmetric function g(r)

$$\nabla^2 g(r) \equiv \frac{d^2g}{dr^2} + \frac{1}{r}\frac{dg}{dr}$$

$$= \kappa^2\Big(\frac{d^2g}{dz^2} + \frac{1}{z}\frac{dg}{dz} \Big)$$

and from the properties of Kelvin functions, [Ref 133]it can be deduced that

$$\nabla^2 K_{ei}(z) = \kappa^2 K_{er}(z)$$

using equation (4.102) it can be shown that

$$\nabla^2 K_{eim}(z) = \kappa^2\Big[\ K_{er}(z) + \log(\tfrac{z}{2})\Big]$$

Hence, another modified Kelvin function can be defined as follows

$$K_{erm}(z) = K_{er}(z) + \log(\tfrac{z}{2}) \tag{4.104}$$

with the following property:

$$\nabla^2 K_{eim}(z) = \kappa^2 K_{erm}(z) \tag{4.105}$$

Notice, that although $K_{er}(z)$ is singular, at $z \rightarrow 0$, $K_{erm}(z)$ has a finite value.

Other properties of modified Kelvin functions can be deduced, and listed as follows:

$$K'_{eim}(z) = K'_{ei}(z) + \frac{z}{2}(\log(\frac{z}{2}) - \frac{1}{2}) \tag{4.106}$$

$$K'_{erm}(z) = K'_{er}(z) + \frac{1}{z} \tag{4.107}$$

$$K''_{eim}(z) = - \frac{1}{z}K'_{eim}(z) + K_{erm}(z) \tag{4.108}$$

$$K''_{eim}(z) = - \frac{1}{z} K'_{erm}(z) - K_{eim} + \frac{z^2}{4}(\log(\frac{z}{2}) - 1) \tag{4.109}$$

where

$$F'(z) = \frac{dF}{dz}; \qquad F''(z) = \frac{d^2F}{dz^2}$$

4.3.3 Modified fundamental solution parameters

From section 4.3.2, w^* can be redefined as follows:

$$w^* = w_1^* + w_2^* \tag{4.110}$$

where

$$w_1^* = \left(e_x\frac{\partial}{\partial x} + e_y\frac{\partial}{\partial y} + e_z\right)\frac{f_1}{D} \tag{4.111}$$

$$w_2^* = \left(e_x\frac{\partial}{\partial x} + e_y\frac{\partial}{\partial y} + e_z\right)\frac{f_2}{D} \tag{4.112}$$

Hence from equations (4.75), (4.76), the fundamental solution parameters, may be redefined as follows:

$$\begin{bmatrix} w^* \\ \frac{\partial w^*}{\partial n} \\ \frac{\partial w^*}{\partial t} \end{bmatrix} = (\underline{U}^* + \underline{U}^{**}) \begin{bmatrix} e_z \\ e_x \\ e_y \end{bmatrix} \tag{4.113}$$

$$\begin{bmatrix} Q_n^* \\ M_n^* \\ M_{tn}^* \end{bmatrix} = (\underset{\sim}{T}^* + \underset{\sim}{T}^{**}) \begin{bmatrix} e_z \\ e_x \\ e_y \end{bmatrix} \qquad (4.114)$$

and from equation (4.110), (4.80) and (4.81), it can be shown that:

a) $\underset{\sim}{U}^*$, $\underset{\sim}{T}^*$ represent fundamental solution parameters for the case of no foundation, and they are defined as follows

$$\underset{\sim}{U}^* = \frac{1}{D} \begin{bmatrix} f_1 & \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_1}{\partial n} & \frac{\partial^2 f_1}{\partial n \partial x} & \frac{\partial^2 f_1}{\partial n \partial y} \\ \frac{\partial f_1}{\partial t} & \frac{\partial^2 f_1}{\partial t \partial x} & \frac{\partial^2 f_1}{\partial t \partial y} \end{bmatrix} \qquad (4.115)$$

$$\underset{\sim}{T}^* = \begin{bmatrix} F_1^* & \frac{\partial F_1^*}{\partial x} & \frac{\partial F_1^*}{\partial y} \\ F_2^* & \frac{\partial F_2^*}{\partial x} & \frac{\partial F_2^*}{\partial y} \\ F_3^* & \frac{\partial F_3^*}{\partial x} & \frac{\partial F_3^*}{\partial y} \end{bmatrix} \qquad (4.116)$$

where

$$F_1^* = -\frac{\partial}{\partial n}\left(\nabla^2 f_1\right)$$

$$F_2^* = -\left(\frac{\partial^2 f_1}{\partial n^2} + \nu \frac{\partial^2 f_1}{\partial t^2}\right)$$

$$F_3^* = -(1-\nu)\frac{\partial^2 f_1}{\partial n \partial t}$$

b) $\underset{\sim}{U}^{**}$, $\underset{\sim}{T}^{**}$ represent fundamental solution parameters, due to the foundation effect and they are defined as follows:

$$\underset{\sim}{U}^{**} = \frac{1}{D}\begin{bmatrix} f_2 & \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_2}{\partial n} & \frac{\partial^2 f_2}{\partial n \partial x} & \frac{\partial^2 f_2}{\partial n \partial y} \\ \frac{\partial f_2}{\partial t} & \frac{\partial^2 f_2}{\partial t \partial x} & \frac{\partial^2 f_2}{\partial t \partial y} \end{bmatrix} \tag{4.117}$$

$$\underset{\sim}{T}^{**} = \begin{bmatrix} F_1^{**} & \frac{\partial F_1^{**}}{\partial x} & \frac{\partial F_1^{**}}{\partial y} \\ F_2^{**} & \frac{\partial F_2^{**}}{\partial x} & \frac{\partial F_2^{**}}{\partial y} \\ F_3^{**} & \frac{\partial F_3^{**}}{\partial x} & \frac{\partial F_3^{**}}{\partial y} \end{bmatrix} \tag{4.118}$$

where

$$F_1^{**} = -\frac{\partial}{\partial n}\left(\nabla^2 f_2\right)$$

$$F_2^{**} = -\left(\frac{\partial^2 f_2}{\partial n^2} + \nu\frac{\partial^2 f_2}{\partial t^2}\right)$$

$$F_3^{**} = -(1-\nu)\frac{\partial^2 f_2}{\partial n \partial t}$$

Derivation of $\underset{\sim}{U}^{*}$, $\underset{\sim}{T}^{*}$ can be found in [Ref 90] and they are listed in Appendix E. The matrices $\underset{\sim}{U}^{**}$, $\underset{\sim}{T}^{**}$ have been derived and they are listed in Tables 4.3, 4.4 respectively.

4.3.4 Modified domain loading terms

The domain loading terms are defined as follows

$$L_\alpha = L_\alpha^{*} + L_\alpha^{**} \tag{4.119}$$

where

$$L_\alpha^{*} = \iint_\Omega q U_{1\alpha}^{*} dx\, dy \tag{4.120}$$

$$L_\alpha^{**} = \iint_\Omega q U_{1\alpha}^{**}\, dx\, dy \tag{4.121}$$

for $\alpha = 1, 2, 3$

The terms L_{α}^{*} can be dealt with in a way similar to that used for thin plates [Ref 90]. The other terms can now be rewritten as follows:

$$L_1^{**} = \iint_{\Omega} qU_{11}^{**}\, dx\, dy \;=\; \iint_{\Omega} \frac{q}{D}\, f_2\, dx\, dy \tag{4.122a}$$

$$L_2^{**} = \iint_{\Omega} qU_{12}^{**}\, dx\, dy \;=\; \iint_{\Omega} \frac{q}{D}\, \frac{\partial f_2}{\partial x}\, dx\, dy \tag{4.122b}$$

$$L_3^{**} = \iint_{\Omega} qU_{13}^{**}\, dx\, dy \;=\; \iint_{\Omega} \frac{q}{D}\, \frac{\partial f_3}{\partial y}\, dx\, dy \tag{4.122c}$$

where

$$f_2 = -\frac{1}{2\pi\kappa^2} K_{eim}(z)$$

Defining a function Q_2, such that

$$f_2 = \nabla^2 Q_2 \tag{4.123}$$

it can be deduced that

$$Q_2 = \frac{1}{2\pi\kappa^2}\left\{ K_{erm}(z) - \frac{z^4}{128}(2\log(\tfrac{z}{2}) - 3)\right\} \tag{4.124}$$

Hence, using integration by parts theorem, it can be deduced, for constant or linear q, that:

$$L_1^{**} = \frac{1}{D}\left\{\oint_{\Gamma} q\frac{\partial Q_2}{\partial n}\, ds \;-\; \oint_{\Gamma} \frac{\partial q}{\partial n} Q_2\, ds\right\} \tag{4.125a}$$

$$L_2^{**} = \frac{1}{D}\left\{\oint_{\Gamma} q\frac{\partial}{\partial x}\left(\frac{\partial Q_2}{\partial n}\right) ds \;-\; \oint_{\Gamma} \frac{\partial q}{\partial n}\frac{\partial Q_2}{\partial x}\, ds\right\} \tag{4.125b}$$

$$L_3^{**} = \frac{1}{D}\left\{\oint_{\Gamma} q\frac{\partial}{\partial y}\left(\frac{\partial Q_2}{\partial n}\right) ds \;-\; \oint_{\Gamma} \frac{\partial q}{\partial n}\frac{\partial Q_2}{\partial y}\, ds\right\} \tag{4.125c}$$

and for constant q, $\frac{\partial q}{\partial n} = 0$. Notice that for

$$f_1 = \frac{1}{2\pi\kappa^2}\left\{\frac{z^2}{4}(\log(\frac{z}{2}) - 1)\right\} = \nabla^2 Q_1 \tag{4.126}$$

it can be shown that

$$Q_1 = \frac{1}{2\pi\kappa^4}\left\{\frac{z^4}{128}(\ 2\log(\frac{z}{2}) - 3\ )\right\} \tag{4.127}$$

Hence, for

$$f = f_1 + f_2 = \nabla^2 Q_m \tag{4.128}$$

then

$$Q_m = Q_1 + Q_2 \tag{4.129}$$

which leads to

$$Q_m = \frac{1}{2\pi\kappa^4} K_{erm}(z) \tag{4.130}$$

and one can write directly (for constant or linear q):

$$L_1 = \frac{1}{D}\left\{\oint_\Gamma q\frac{\partial Q_m}{\partial n}\, ds - \oint_\Gamma \frac{\partial q}{\partial n} Q_m\, ds\right\} \tag{4.131a}$$

$$L_2 = \frac{1}{D}\left\{\oint_\Gamma q\frac{\partial}{\partial x}\left(\frac{\partial Q_m}{\partial n}\right) ds - \oint_\Gamma \frac{\partial q}{\partial n}\frac{\partial Q_m}{\partial x}\, ds\right\} \tag{4.131b}$$

$$L_3 = \frac{1}{D}\left\{\oint_\Gamma q\frac{\partial}{\partial y}\left(\frac{\partial Q_m}{\partial n}\right) ds - \oint_\Gamma \frac{\partial q}{\partial n}\frac{\partial Q_m}{\partial y}\, ds\right\} \tag{4.131c}$$

Due to a hidden singularity, if Q (defined by equation 4.85) is used in equations (4.131) instead of Q_m, wrong answers will be obtained. This fact has been recognised by some authors [Ref 115,120], who tried to provide alternative solutions.

Equation (4.131) may be rewritten as follows:

$$L_\alpha(x_i,y_i) = \oint_\Gamma q p^*_\alpha \, ds - \oint_\Gamma \frac{\partial q}{\partial n} q^*_\alpha \, ds \tag{4.132}$$

where

$$\alpha = 1, 2, 3$$

and the fundamental solution parameters p^*_α, q^*_α have been derived and are listed in Table 4.5

4.3.5 Boundary element equations

Using constant boundary elements, boundary integral equations at a source point can be descretised, as described in section 4.2.6. For source points being taken at all boundary nodes, the following system of equations is produced:

$$\hat{\underset{\sim}{H}} \, \underset{\sim}{\delta} = \underset{\sim}{G} \, \underset{\sim}{t} + \underset{\sim}{l} \tag{4.133}$$

where

$$\hat{\underset{\sim}{H}} = \underset{\sim}{C} + H^* + \underset{\sim}{H}^{**} \tag{4.134}$$

$$\underset{\sim}{G} = \underset{\sim}{G}^* + G^{**} \tag{4.135}$$

and

$$H^*_{3(i-1)+\alpha,3(j-1)+\beta} = \int_j T^*_{\beta,\alpha}(x - x_i, y - y_i) \, ds \tag{4.136}$$

$$H^{**}_{3(i-1)+\alpha,3(j-1)+\beta} = \int_j T^{**}_{\beta,\alpha}(x - x_i, y - y_i) \, ds \tag{4.137}$$

$$G^*_{3(i-1)+\alpha,3(j-1)+\beta} = \int_j U^*_{\beta,\alpha}(x - x_i, y - y_i) \, ds \tag{4.138}$$

$$G^{**}_{3(i-1)+\alpha,3(j-1)+\beta} = \int_j U^{**}_{\beta,\alpha}(x - x_i, y - y_i) \, ds \tag{4.139}$$

$$l_{3(i-1)+\alpha} = L_\alpha(x_i, y_i)$$

$$= \sum_{j=1}^{m} \left\{ \int_j q\, p^{*}_\alpha\, ds - \int_j \frac{\partial q}{\partial n}\, q^{*}_\alpha\, ds \right\} \qquad (4.140)$$

The boundary-element standard procedure can be carried out on matrix equation (4.133).

4.3.6 Analysis of singular integral terms

Notice that for $K \rightarrow 0$, $\underline{H}^{**}$, $\underline{G}^{**}$ defined by equations (4.137), (4.139) should vanish leading to a system of equations for plate bending with no foundation. This fact has been used in this work to calculate the divergent integrals in $\underline{H}^{*}$ by means of rigid translation and rotation conditions applicable to the case without foundation. Other singular integrals in $\underline{H}^{*}$, $\underline{G}^{*}$ can be estimated by means of analytical expressions, in a way similar to that used for plate-bending problems [Ref 90]. The remaining task is to analyse the singular terms in $\underline{H}^{**}$, $\underline{G}^{**}$.

From the definition of modified Kelvin functions, it can be proved that:

i) $\lim_{z \rightarrow 0} K_{erm}(z) = -\gamma$

ii) $\lim_{z \rightarrow 0} K_{eim}(z) = 0$

iii) $\lim_{z \rightarrow 0} K'_{erm}(z) = 0$

iv) $\lim_{z \rightarrow 0} K'_{eim}(z) = 0$

v) $\lim_{z \rightarrow 0} \frac{K'_{erm}(z)}{z} = -\frac{\pi}{8}$

vi) $\lim_{z \rightarrow 0} \frac{K'_{eim}(z)}{z} = -\frac{\gamma}{2}$

vii) $\lim_{z \rightarrow 0} A_m(z) = 0$

viii) $\lim_{z \rightarrow 0} B_m(z) = \frac{\pi}{4}$

Hence, there is no singularity in all terms of $\underline{H}^{**}$, $\underline{G}^{**}$, $\underline{l}$ defined by equations (4.137), (4.139) and (4.140), and this may prove a major advantage for the derivations based upon modified Kelvin functions. Another advantage, which has been observed when validating the the developed programs. When elastic foundation stiffness "K" is very small or goes to zero the old derivations lead to divergent answers. The program based upon modified derivation can have a parameter to switch off the calculations of $\underline{H}^{**}$, $\underline{G}^{**}$ when $K \rightarrow 0$, and it will not lead to divergent answers for small values of K.

$$U_{11} = f = -\frac{1}{2\pi D\kappa^2}K_{ei}(z)$$

$$U_{12} = \frac{\partial f}{\partial x} = -\frac{1}{2\pi D\kappa}\frac{\partial r}{\partial x}K'_{ei}(z)$$

$$U_{13} = \frac{\partial f}{\partial y} = -\frac{1}{2\pi D\kappa}\frac{\partial r}{\partial y}K'_{ei}(z)$$

$$U_{21} = \frac{\partial f}{\partial n} = -\frac{1}{2\pi D\kappa}\frac{\partial r}{\partial n}K'_{ei}(z)$$

$$U_{22} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial n}\right) = -\frac{1}{2\pi D}\left\{\frac{\hat{i}\cdot\hat{n}}{z}K'_{ei}(z) + \frac{\partial r}{\partial x}\frac{\partial r}{\partial n}A(z)\right\}$$

$$U_{23} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial n}\right) = -\frac{1}{2\pi D}\left\{\frac{\hat{j}\cdot\hat{n}}{z}K'_{ei}(z) + \frac{\partial r}{\partial y}\frac{\partial r}{\partial n}A(z)\right\}$$

$$U_{31} = \frac{\partial f}{\partial t} = -\frac{1}{2\pi D\kappa}\frac{\partial r}{\partial t}K'_{ei}(z)$$

$$U_{32} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right) = -\frac{1}{2\pi D}\left\{\frac{\hat{i}\cdot\hat{t}}{z}K'_{ei}(z) + \frac{\partial r}{\partial x}\frac{\partial r}{\partial t}A(z)\right\}$$

$$U_{33} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial t}\right) = -\frac{1}{2\pi D}\left\{\frac{\hat{j}\cdot\hat{t}}{z}K'_{ei}(z) + \frac{\partial r}{\partial y}\frac{\partial r}{\partial t}A(z)\right\}$$

$$r = \sqrt{(x - x_i)^2 + (y - y_i)^2}$$

$$z = \kappa r$$

$$\kappa = \left(\frac{K}{D}\right)^{\frac{1}{4}}$$

$$A(z) = K_{er}(z) - \frac{2}{z}K'_{ei}(z)$$

[Table 4.1] $\underline{U}$ parameters

$$T_{11} = F_1 = \frac{\kappa}{2\pi}\frac{\partial r}{\partial n}K'_{er}(z)$$

$$T_{12} = \frac{\partial}{\partial x}F_1 = \frac{\kappa^2}{2\pi}\left\{\frac{\hat{\imath}\cdot\hat{n}}{z}K'_{er}(z) - \frac{\partial r}{\partial x}\frac{\partial r}{\partial n}B(z)\right\}$$

$$T_{13} = \frac{\partial}{\partial y}F_1 = \frac{\kappa^2}{2\pi}\left\{\frac{\hat{\jmath}\cdot\hat{n}}{z}K'_{er}(z) - \frac{\partial r}{\partial y}\frac{\partial r}{\partial n}B(z)\right\}$$

$$T_{21} = F_2 = \frac{1}{4\pi}\left\{(1+\nu)K_{er}(z) + (1-\nu)\left(2\left(\frac{\partial r}{\partial n}\right)^2 - 1\right)A(z)\right\}$$

$$T_{22} = \frac{\partial}{\partial x}F_2 = \frac{\kappa}{2\pi}\left\{K'_{er}(z)\left(\nu\frac{\partial r}{\partial x} + (1-\nu)\frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial n}\right)^2\right) + \frac{(1-\nu)A(z)}{z}\left(2(\hat{\imath}\cdot\hat{n})\frac{\partial r}{\partial n} + \frac{\partial r}{\partial x} - 4\frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial n}\right)^2\right)\right\}$$

$$T_{23} = \frac{\partial}{\partial y}F_2 = \frac{\kappa}{2\pi}\left\{K'_{er}(z)\left(\nu\frac{\partial r}{\partial y} + (1-\nu)\frac{\partial r}{\partial y}\left(\frac{\partial r}{\partial n}\right)^2\right) + \frac{(1-\nu)A(z)}{z}\left(2(\hat{\jmath}\cdot\hat{n})\frac{\partial r}{\partial n} + \frac{\partial r}{\partial y} - 4\frac{\partial r}{\partial y}\left(\frac{\partial r}{\partial n}\right)^2\right)\right\}$$

$$T_{31} = F_3 = \frac{(1-\nu)}{2\pi}\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}A(z)$$

$$T_{32} = \frac{\partial}{\partial x}F_3 = \frac{(1-\nu)\kappa}{2\pi}\left\{\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}\frac{\partial r}{\partial x}K'_{er}(z) + \frac{A(z)}{z}\left((\hat{\imath}\cdot\hat{n})\frac{\partial r}{\partial n} + (\hat{\imath}\cdot\hat{\imath})\frac{\partial r}{\partial t} - 4\frac{\partial r}{\partial x}\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}\right)\right\}$$

$$T_{33} = \frac{\partial}{\partial y}F_3 = \frac{(1-\nu)\kappa}{2\pi}\left\{\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}\frac{\partial r}{\partial y}K'_{er}(z) + \frac{A(z)}{z}\left((\hat{\jmath}\cdot\hat{n})\frac{\partial r}{\partial n} + (\hat{\jmath}\cdot\hat{\imath})\frac{\partial r}{\partial t} - 4\frac{\partial r}{\partial y}\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}\right)\right\}$$

where

$$B(z) = K_{er}(z) + \frac{2}{z}K'_{ei}(z)$$

[Table 4.2] $\underset{\sim}{T}$ Parameters

$$U^{**}_{11} = f_2 = -\frac{1}{2\pi D\kappa}K_{eim}(z)$$

$$U^{**}_{12} = \frac{\partial f_2}{\partial x} = -\frac{1}{2\pi D\kappa}\frac{\partial r}{\partial x}K'_{eim}(z)$$

$$U^{**}_{13} = \frac{\partial f_2}{\partial y} = -\frac{1}{2\pi D\kappa}\frac{\partial r}{\partial y}K'_{eim}(z)$$

$$U^{**}_{21} = \frac{\partial f_2}{\partial n} = -\frac{1}{2\pi D\kappa}\frac{\partial r}{\partial n}K'_{eim}(z)$$

$$U^{**}_{22} = -\frac{1}{2\pi D}\left\{\frac{\hat{n}\cdot\hat{\imath}}{z}\,K'_{eim}(z) + \frac{\partial r}{\partial x}\frac{\partial r}{\partial n}\,A_m(z)\right\}$$

$$U^{**}_{23} = -\frac{1}{2\pi D}\left\{\frac{\hat{\jmath}\cdot\hat{n}}{z}\,K'_{eim}(z) + \frac{\partial r}{\partial y}\frac{\partial r}{\partial n}\,A_m(z)\right\}$$

$$U^{**}_{31} = \frac{\partial f_2}{\partial t} = -\frac{1}{2\pi D\kappa}\frac{\partial r}{\partial t}\,K'_{eim}(z)$$

$$U^{**}_{32} = -\frac{1}{2\pi D}\left\{\frac{\hat{\imath}\cdot\hat{\imath}}{z}\,K'_{eim}(z) + \frac{\partial r}{\partial x}\frac{\partial r}{\partial t}\,A_m(z)\right\}$$

$$U^{**}_{33} = -\frac{1}{2\pi D}\left\{\frac{\hat{\jmath}\cdot\hat{\imath}}{z}\,K'_{eim}(z) + \frac{\partial r}{\partial y}\frac{\partial r}{\partial t}\,A_m(z)\right\}$$

$$A_m(z) = K_{erm}(z) - \frac{2}{z}\,K'_{eim}(z)$$

[Table 4.3] $\underline{U}^{**}$ parameters

$$T_{11}^{**} = F_1^{**} = \frac{\kappa}{2\pi}\frac{\partial r}{\partial n}K'_{erm}(z)$$

$$T_{12}^{**} = \frac{\partial}{\partial x}F_1^{**} = \frac{\kappa^2}{2\pi}\left\{\frac{\hat{i}\cdot\hat{n}}{z}\,K'_{erm}(z) - \frac{\partial r}{\partial x}\frac{\partial r}{\partial n}\,B_m(z)\right\}$$

$$T_{13}^{**} = \frac{\partial}{\partial y}F_1^{**} = \frac{\kappa^2}{2\pi}\left\{\frac{\hat{j}\cdot\hat{n}}{z}\,K'_{erm}(z) - \frac{\partial r}{\partial y}\frac{\partial r}{\partial n}\,B_m(z)\right\}$$

$$T_{21}^{**} = F_2^{**} = \frac{1}{4\pi}\left\{(1+\nu)\,K_{erm}(z) + (1-\nu)\left(2\left(\frac{\partial r}{\partial n}\right)^2 - 1\right)A(z)\right\}$$

$$T_{22}^{**} = \frac{\partial}{\partial x}F_2^{**} = \frac{\kappa}{2\pi}\left\{K'_{erm}(z)\left(\nu\frac{\partial r}{\partial x} + (1-\nu)\frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial n}\right)^2\right) + \frac{(1-\nu)A_m(z)}{z}\left(2(\hat{i}\cdot\hat{n})\frac{\partial r}{\partial n} + \frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial n}\right)^2\right)\right\}$$

$$T_{23}^{**} = \frac{\partial}{\partial y}F_2^{**} = \frac{\kappa}{2\pi}\left\{K'_{erm}(z)\left(\nu\frac{\partial r}{\partial y} + (1-\nu)\frac{\partial r}{\partial y}\left(\frac{\partial r}{\partial n}\right)^2\right) + \frac{(1-\nu)A_m(z)}{z}\left(2(\hat{j}\cdot\hat{n})\frac{\partial r}{\partial n} + \frac{\partial r}{\partial y} - 4\frac{\partial r}{\partial y}\left(\frac{\partial r}{\partial n}\right)^2\right)\right\}$$

$$T_{31}^{**} = F_3^{**} = \frac{(1-\nu)}{2\pi}\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}A_m(z)$$

$$T_{32}^{**} = \frac{\partial}{\partial x}F_3^{**} = \frac{(1-\nu)\kappa}{2\pi}\left\{\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}\frac{\partial r}{\partial x}K'_{erm}(z) + \frac{A_m(z)}{z}\left((\hat{i}\cdot\hat{n})\frac{\partial r}{\partial n} + (\hat{i}\cdot\hat{\imath})\frac{\partial r}{\partial t} - 4\frac{\partial r}{\partial x}\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}\right)\right\}$$

$$T_{33}^{**} = \frac{\partial}{\partial y}F_3^{**} = \frac{(1-\nu)\kappa}{2\pi}\left\{\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}\frac{\partial r}{\partial y}K'_{erm}(z) + \frac{A_m(z)}{z}\left((\hat{j}\cdot\hat{n})\frac{\partial r}{\partial n} + (\hat{j}\cdot\hat{\imath})\frac{\partial r}{\partial t} - 4\frac{\partial r}{\partial y}\frac{\partial r}{\partial n}\frac{\partial r}{\partial t}\right)\right\}$$

$$B_m(z) = K'_{erm}(z) + \frac{2}{z}\,K'(z)$$

Table 4.4 $\underline{T}^{**}$ Parameters

$$p_1^* = \frac{1}{2\pi D\kappa^3}\frac{\partial r}{\partial n}\,K'_{erm}(z)$$

$$p_2^* = \frac{1}{2\pi D\kappa^2}\left\{\frac{\hat{i}\cdot\hat{n}}{z}\,K'_{erm}(z) - \frac{\partial r}{\partial x}\frac{\partial r}{\partial n}\,B_p(z)\right\}$$

$$p_3^* = \frac{1}{2\pi D\kappa^2}\left\{\frac{\hat{j}\cdot\hat{n}}{z}\,K'_{erm}(z) - \frac{\partial r}{\partial y}\frac{\partial r}{\partial n}\,B_p(z)\right\}$$

$$q_1^* = \frac{1}{2\pi D\kappa^4}\,K_{erm}(z)$$

$$q_2^* = \frac{1}{2\pi D\kappa^3}\frac{\partial r}{\partial x}\,K'_{erm}(z)$$

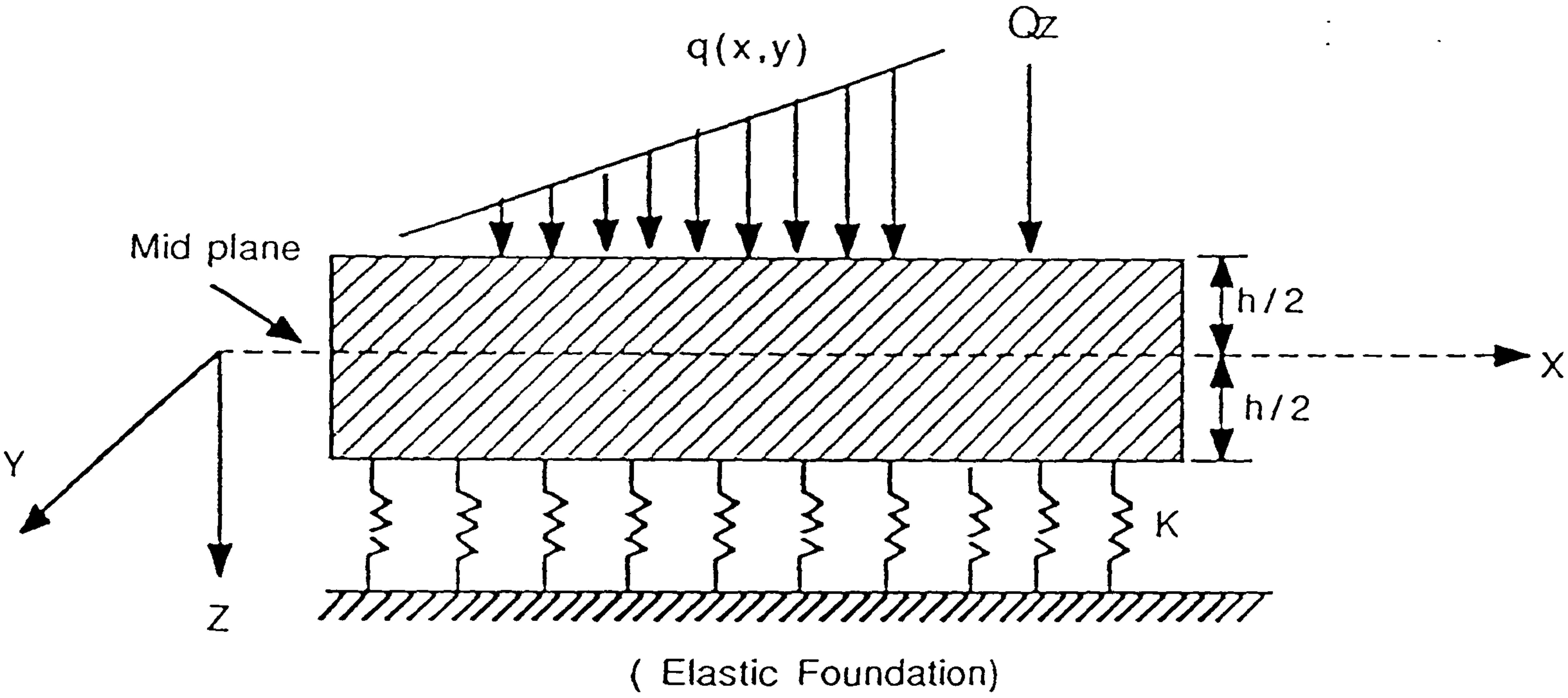
$$q_3^* = \frac{1}{2\pi D\kappa^3}\frac{\partial r}{\partial y}\,K'_{erm}(z)$$

where

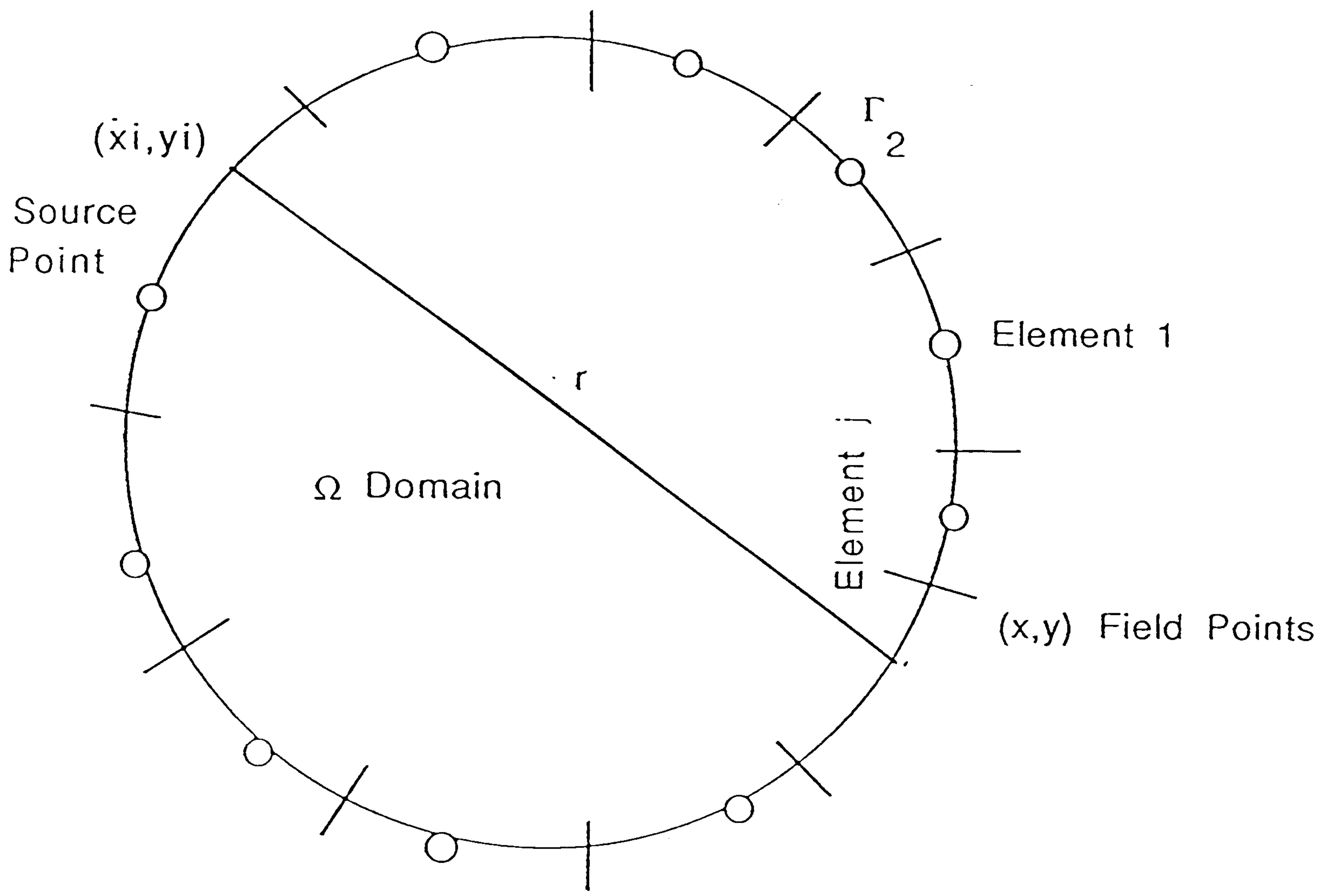
$$z = \kappa r$$

$$B_p(z) = K_{ei}(z) + \frac{2}{z}K'_{erm}(z)$$

[Table 4.5] Fundamental Solution Parameters for domain loading terms.



(Fig 4.1) Section of a plate on an Elastic Foundation



(Fig 4.2) Discretised Boundary

CHAPTER FIVE

BOUNDARY ELEMENT ANALYSIS OF THICK PLATES ON ELASTIC FOUNDATION

5.1 INTRODUCTION

In this chapter, the boundary element analysis of thick plates on elastic foundation is presented. New fundamental solutions for different possible cases are derived using Fourier and Hankel integral transforms. Details of analysis of loading terms in terms of boundary integral terms are also introduced. The fundamental solution parameters contain singular terms which may lead to divergent integrals. Hence, a method which is based upon a fictitious boundary is suggested and it does not contain any singularity whatsoever.

An alternative approach based upon the fundamental solution of thin plates on elastic foundation is also presented. Most of the work introduced in this chapter has been derived for the first time by the author and has been employed in computer programs as reviewed in the next chapter.

5.2 BASIC GOVERNING EQUATIONS

5.2.1 Transverse stresses

For the analysis of thick plates on elastic Winkler foundation, the parameters defined by equations (4.1) to (4.8) are used. The boundary conditions for a thick plate (as shown in fig 4.1), are the same as given by equations (4.9) and (4.10). The governing equations of equilibrium, over the plate thickness at any point (x,y) are defined by equations (4.16). Transverse stresses will be considered similar to those given by equations (4.18) and (4.20),
i.e.

$$\tau_{xz} = \frac{3}{2}\frac{Q_x}{h}\left(1 - \frac{4z^2}{h^2}\right) \tag{5.1a}$$

$$\tau_{yz} = \frac{3}{2}\frac{Q_y}{h}\left(1 - \frac{4z^2}{h^2}\right) \tag{5.1b}$$

and

$$\sigma_z = -\frac{(q+Kw)}{2} + \frac{(q-Kw)}{2h}\left(3z - \frac{4z^3}{h^2}\right) \tag{5.2}$$

For linear elastic isotropic materials, the transverse shear strains can be derived from equation (5.1) as follows:

$$\gamma_{xy}=\frac{\tau_{xz}}{G}=\frac{3}{2}\,\frac{Q_x}{hG}\,(\,1-\frac{4z^2}{h^2}\,) \tag{5.3a}$$

$$\gamma_{yz}=\frac{\tau_{yz}}{G}=\frac{3}{2}\,\frac{Q_y}{hG}\,(\,1-\frac{4z^2}{h^2}\,) \tag{5.3b}$$

and it is clear that at z=0

$$\gamma^o_{xz}=\frac{3Q_x}{2hG};\qquad\qquad \gamma^o_{yz}=\frac{3Q_y}{2hG} \tag{5.4}$$

5.2.2 <u>Displacement, Strain and Stress Components.</u>

Using an approach similar to that used in section 3.4.1, it can be deduced from equations (3.70) and (5.3) that the displacement components at any point (x,y,z) inside the plate maybe approximated as follows:

$$u(x,y,z)\simeq u_o(x,y)-z\,\frac{\partial w}{\partial x}+\frac{3}{2}\,\frac{Q_x}{hG}\,(\,z-\frac{4z^3}{h^2}\,) \tag{5.5a}$$

$$v(x,y,z)\simeq v_o(x,y)-z\,\frac{\partial w}{\partial y}+\frac{3}{2}\,\frac{Q_y}{hG}\,(\,z-\frac{4z^3}{h^2}\,) \tag{5.5b}$$

$$w(x,y,z)\simeq -\,w(x,y) \tag{5.5c}$$

and it is clear from plate bending theory, that $u_o(x,y)$ and $v_o(x,y)$ will only be due to in-plane loading, which will not be considered in this analysis, i.e. for the case of out-of-plane loading, the displacement components may be approximated as follows:

$$u(x,y,z)\simeq -\,z\,\frac{\partial w}{\partial x}+\frac{h^2}{4}\,\frac{Q_x}{(1-\nu)D}\,(\,z-\frac{4z^3}{h^2}\,) \tag{5.6a}$$

$$v(x,y,z)\simeq -\,z\,\frac{\partial w}{\partial y}+\frac{h^2}{4}\,\frac{Q_y}{(1-\nu)D}\,(\,z-\frac{4z^3}{h^2}\,) \tag{5.6b}$$

$$w(x,y,z)\simeq -\,w(x,y) \tag{5.6c}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

Using Cauchy's strain-displacement relationships, the strain components can be expressed in terms of the displacement components defined by means of equations (5.6), as follows:

$$\epsilon_x = -z\frac{\partial^2 w}{\partial x^2} + \frac{h^2}{4}\frac{1}{(1-\nu)D}\frac{\partial Q_x}{\partial x}\left(z - \frac{4z^3}{3h^2}\right) \tag{5.7a}$$

$$\epsilon_y = -z\frac{\partial^2 w}{\partial y^2} + \frac{h^2}{4}\frac{1}{(1-\nu)D}\frac{\partial Q_y}{\partial y}\left(z - \frac{4z^3}{3h^2}\right) \tag{5.7b}$$

$$\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x\partial y} + \frac{h^2}{4}\frac{1}{(1-\nu)D}\left(\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x}\right)\left(z - \frac{4z^3}{3h^2}\right) \tag{5.7c}$$

The transverse stress components are defined in equations (5.1), (5.2), and the other stress components can be expressed in terms of ϵ_x, ϵ_y, γ_{xy}, σ_z as shown in equation (4.23), from which and by using equations (5.7), it can be shown that:

$$\sigma_x = -\frac{12D}{h^3} z\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + \frac{3}{(1-\nu)\,h}\left(z - \frac{4z^3}{3h^2}\right)\left(\frac{\partial Q_x}{\partial x} + \nu\frac{\partial Q_y}{\partial y}\right)$$

$$+ \frac{3\nu}{h(1-\nu)}\left[\frac{-(q + K\,w)}{2} + \frac{(q - K\,w)}{2}\left(z - \frac{4z^3}{3h^2}\right)\right] \tag{5.8a}$$

$$\sigma_y = -\frac{12\,D}{h^3} z\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) + \frac{3}{(1-\nu)h}\left(z - \frac{4\,z^3}{3h^2}\right)\left(\nu\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right)$$

$$+ \frac{3\nu}{h(1-\nu)}\left[\frac{-(q + K\,w)}{2} + \frac{(q - K\,w)}{2}\left(z - \frac{4z^3}{3h^2}\right)\right] \tag{5.8b}$$

$$\tau_{xy} = -(1-\nu)\frac{12D}{h^3} z\frac{\partial^2 w}{\partial x\partial y} + \frac{3}{(1-\nu)\,h}\left(z - \frac{4z^3}{3h^2}\right)\frac{(1-\nu)}{2}\left(\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x}\right) \tag{5.8c}$$

5.2.3 Moments and shear forces

From the definition of M_x, M_y, M_{xy} and using equations (5.8), it can be proved that:

$$M_x = \int_{-h/2}^{h/2} z\sigma_x \, dz = - D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)$$

$$+ \frac{1}{(1-\nu)\lambda^2} \left[2\left(\frac{\partial Q_x}{\partial x} + \nu \frac{\partial Q_y}{\partial y} \right) + \nu(q - K\, w) \right] \tag{5.9a}$$

$$M_y = \int_{-h/2}^{h/2} z\sigma_y \, dz = - D \left(\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)$$

$$+ \frac{1}{(1-\nu)\lambda^2} \left[2\left(\nu \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right) + \nu(q - K\, w) \right] \tag{5.9b}$$

$$M_{xy} = -(1 - \nu)\, D \frac{\partial^2 w}{\partial x \partial y} + \frac{1}{\lambda^2}\left(\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} \right) \tag{5.9c}$$

where

$$\lambda^2 = \frac{10}{h^2} \tag{5.10}$$

Equation (5.9) may be rewritten as follows

$$M_x = D \left(\frac{\partial \theta_x}{\partial x} + \nu \frac{\partial \theta_y}{\partial y} \right) + \frac{\nu}{(1-\nu)\lambda^2}(q - K\, w) \tag{5.11a}$$

$$M_y = D \left(\nu \frac{\partial \theta_x}{\partial x} + \frac{\partial \theta_y}{\partial y} \right) + \frac{\nu}{(1-\nu)\lambda^2}(q - K\, w) \tag{5.11b}$$

$$M_{xy} = (1 - \nu)\frac{D}{2}\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) \tag{5.11c}$$

where the parameters θ_x and θ_y are defined as follows:

$$\theta_x = \frac{2Q_x}{D(1-\nu)\lambda^2} - \frac{\partial w}{\partial x} \tag{5.12a}$$

$$\theta_y=\frac{2Q_y}{D(1-\nu)\lambda^2} - \frac{\partial w}{\partial y} \tag{5.12b}$$

i.e. and from equation (5.12) we have the following:

$$Q_x = \frac{D(1-\nu)\lambda^2}{2}(\theta_x + \frac{\partial w}{\partial x}) \tag{5.13a}$$

$$Q_y = \frac{D(1-\nu)\lambda^2}{2}(\theta_y + \frac{\partial w}{\partial y}) \tag{5.13b}$$

using average values of transverse shear stresses and strains, as given in section 3.3.1, it can be proved that:

$$\bar{\gamma}_{xz} = \frac{\bar{\tau}_{xz}}{\frac{5}{6}G} = \frac{2Q_x}{D(1-\nu)\lambda^2} \tag{5.14a}$$

and

$$\bar{\gamma}_{yz} = \frac{\bar{\tau}_{yz}}{\frac{5}{6}G} = \frac{2Q_y}{D(1-\nu)\lambda^2} \tag{5.14b}$$

i.e.

$$\theta_x=\bar{\gamma}_{xz}-\frac{\partial w}{\partial x} \tag{5.15a}$$

$$\theta_y=\bar{\gamma}_{yz}-\frac{\partial w}{\partial y} \tag{5.15b}$$

which are compatible with the average slope angles used for the derivation of first order shear elements as defined by equation (3.33), (in which an engineering notation was adopted). However, the boundary element method (BEM) derivation presented here is of a higher order.

5.2.4 Derivation of weighted residual expressions

Considering an approximate solution which satisfies the given boundary condition, a weighted-residual expression can be deduced from equilibrium equations (4.16), in terms of the three weighting functions θ_x^*, θ_y^* and w^* as given by equation (4.30), which can be integrated by parts as shown in equation (4.32), i.e.

$$\oint_{\Gamma}\left(\theta_x^* \, t_x + \theta_y^* \, t_y + w^* t_z\right) ds \; + \iint_{\Omega} w^*(q\text{-}Kw) \, dx \, dy$$

$$-\iint_{\Omega}\left\{M_x \frac{\partial\theta_x^*}{\partial x} + M_{xy}\left(\frac{\partial\theta_x^*}{\partial y} + \frac{\partial\theta_y^*}{\partial x}\right) + M_y\frac{\partial\theta_y^*}{\partial y}\right\} dx \, dy$$

$$-\iint_{\Omega}\left\{Q_x\left(\theta_x^* + \frac{\partial w^*}{\partial x}\right) + Q_y\left(\theta_y^* + \frac{\partial w^*}{\partial y}\right)\right\} dx \, dy = 0 \tag{5.16}$$

Defining the following parameters:

$$M_x^* = D\left(\frac{\partial\theta_x^*}{\partial x} + \nu\,\frac{\partial\theta_y^*}{\partial y}\right) \tag{5.17a}$$

$$M_y^* = D\left(\nu\,\frac{\partial\theta_x^*}{\partial x} + \frac{\partial\theta_y^*}{\partial y}\right) \tag{5.17b}$$

$$M_{xy}^* = (1 - \nu)\frac{D}{2}\left(\frac{\partial\theta_x^*}{\partial y} + \frac{\partial\theta_y^*}{\partial x}\right) \tag{5.17c}$$

and

$$Q_x^* = \frac{D(1 - \nu)\lambda^2}{2}\left(\theta_x^* + \frac{\partial w^*}{\partial x}\right) \tag{5.18a}$$

$$Q_y^* = \frac{D(1 - \nu)\lambda^2}{2}\left(\theta_y^* + \frac{\partial w^*}{\partial y}\right) \tag{5.18b}$$

and using equation (5.11), (5.13), it can be proved that:

$$M_x \frac{\partial\theta_x^*}{\partial x} + M_{xy}\left(\frac{\partial\theta_x^*}{\partial y} + \frac{\partial\theta_y^*}{\partial x}\right) + M_y\frac{\partial\theta_y^*}{\partial y} = M_x^* \frac{\partial\theta_x}{\partial x} + M_{xy}^*\left(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right) + M_y^*\frac{\partial\theta_y}{\partial y}$$
$$+ \frac{\nu}{(1 - \nu)\lambda^2}(q - Kw)\left(\frac{\partial\theta_x^*}{\partial x} + \frac{\partial\theta_y^*}{\partial y}\right) \tag{5.19}$$

and

$$Q_x\left(\theta_x^* + \frac{\partial w^*}{\partial x}\right) + Q_y\left(\theta_y^* + \frac{\partial w^*}{\partial y}\right) = Q_x^*\left(\theta_x + \frac{\partial w}{\partial x}\right) + Q_y^*\left(\theta_y + \frac{\partial w}{\partial y}\right) \tag{5.20}$$

Substituting from equations (5.19) and (5.20) into expression (5.16), and integrating the result by parts once more, it can be proved that:

$$\oint_{\Gamma}\left(\theta_x^* \, t_x + \theta_y^* \, t_y + w^* t_z\right) ds - \oint_{\Gamma}\left(\theta_x \, t_x^* + \theta_y \, t_y^* + w \, t_z^*\right) ds$$

$$+ \iint_{\Omega} q\left(w^* - \frac{\nu}{(1-\nu)\lambda^2}\left(\frac{\partial \theta_x^*}{\partial x} + \frac{\partial \theta_y^*}{\partial y}\right)\right) dx\, dy$$

$$+ \iint_{\Omega}\left\{\theta_x\left(\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - Q_x^*\right) + \theta_y\left(\frac{\partial M_{xy}^*}{\partial x} + \frac{\partial M_y^*}{\partial y} - Q_y^*\right)\right\} dx\, dy$$

$$+ \iint_{\Omega} w\left\{\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - Kw^* + \frac{K\nu}{(1-\nu)\lambda^2}\left(\frac{\partial \theta_x^*}{\partial x} + \frac{\partial \theta_y^*}{\partial y}\right)\right\} dx\, dy = 0 \qquad (5.21)$$

where

$$t_x^* = l\, M_x^* + m\, M_{xy}^* \qquad (5.22a)$$

$$t_y^* = l\, M_{xy}^* + m\, M_y^* \qquad (5.22b)$$

$$t_z^* = l\, Q_x^* + m\, Q_y^* \qquad (5.22c)$$

5.3 DERIVATION OF FUNDAMENTAL SOLUTION

5.3.1 Strain function method

The fundamental solution parameters θ_x^*, θ_y^*, w^* are defined such that

$$\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - Q_x^* = -\, e_x \delta(x - x_i, y - y_i) \qquad (5.23a)$$

$$\frac{\partial M_{xy}^*}{\partial x} + \frac{\partial M_y^*}{\partial y} - Q_y^* = -\, e_y \delta(x - x_i, y - y_i) \qquad (5.23b)$$

$$\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - Kw^* + \frac{K\nu}{(1-\nu)\lambda^2}\left(\frac{\partial \theta_x^*}{\partial x} + \frac{\partial \theta_y^*}{\partial y}\right) = -\, e_z \delta(x - x_i, y - y_i) \qquad (5.23c)$$

where

e_x, e_y and e_z are arbitrary constants.

and

M_x^*, M_y^*, M_{xy}^*, Q_x^* and Q_y^* are as defined in equation (5.17) and (5.18), respectively.

Substituting from equations (5.17), (5.18) into (5.23), the fundamental governing equations can be expressed in terms of θ_x^*, θ_y^*, w^* as follows:

$$\frac{D}{2}(1-\nu)\,\underset{\sim}{\mathfrak{D}}\begin{bmatrix}\theta_x^*\\ \theta_y^*\\ w^*\end{bmatrix} = -\,\delta(x-x_i\,,\,y-y_i)\begin{bmatrix}e_x\\ e_y\\ e_z\end{bmatrix} \qquad (5.24)$$

where

$$\underset{\sim}{\mathfrak{D}} = \begin{bmatrix} \nabla^2-\lambda^2+\alpha\frac{\partial^2}{\partial x^2} & \alpha\frac{\partial^2}{\partial x\partial y} & -\lambda^2\frac{\partial}{\partial x} \\ \alpha\frac{\partial^2}{\partial x\partial y} & \nabla^2-\lambda^2+\alpha\frac{\partial^2}{\partial x^2} & -\lambda^2\frac{\partial}{\partial y} \\ \gamma\frac{\partial}{\partial x} & \gamma\frac{\partial}{\partial y} & \lambda^2(\nabla^2-\beta) \end{bmatrix} \qquad (5.25)$$

in which

$$\lambda^2 = \frac{10}{h^2}, \qquad \alpha = \frac{(1+\nu)}{(1-\nu)}, \qquad \beta = \frac{2K}{D(1-\nu)\lambda^2}$$

$$\gamma = \lambda^2 + \frac{\nu}{(1-\nu)}\beta, \qquad D=\frac{Eh}{12(1-\nu^2)}$$

Strain functions f_1, f_2 and f_3 may be assumed such that

$$\begin{bmatrix}\theta_x^*\\ \theta_y^*\\ w^*\end{bmatrix} = \underset{\sim}{C}\begin{bmatrix}f_1\\ f_2\\ f_3\end{bmatrix} \qquad (5.26)$$

where

$\underset{\sim}{C}$ is a differential operator matrix.

If $\underset{\sim}{C}$ is selected such that:

$$\underset{\sim}{C}=\frac{1}{\lambda^2}(\underset{\sim}{\mathfrak{D}}^*)^t \qquad (5.27)$$

where $\mathfrak{D}^*$ is the cofactor matrix of $\mathfrak{D}$, then by substituting from (5.27) into (5.24), it can be shown that:

$$\frac{D}{2}(1-\nu)\frac{1}{\lambda^2}|\mathfrak{D}|\, \underline{I}_{3\times 3} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = -\,\delta(x-x_i\,,\,y-y_i) \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \tag{5.28}$$

Note that the determinant of the $\mathfrak{D}$ matrix can be expressed explicitly as follows:

$$|\mathfrak{D}| = \frac{2\lambda^2}{(1-\nu)}(\nabla^2-\lambda^2)(\nabla^4-2b\nabla^2+\kappa^4) \tag{5.29}$$

where

$$b = \frac{(2-\nu)}{2(1-\nu)\lambda^2}\frac{K}{D}$$

and

$$\kappa^4 = \frac{K}{D}$$

and equation (5.28) may be reduced to

$$D(\nabla^2-\lambda^2)(\nabla^4-2b\nabla^2+\kappa^4) \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = -\,\delta(x-x_i\,,\,y-y_i) \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \tag{5.30}$$

which proves that f_1, f_2 and f_3 can be expressed as follows:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = f \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \tag{5.31}$$

and it is clear from equation (5.30) that "f" is governed by the following differential equation:

$$(\nabla^2-\lambda^2)(\nabla^4-2b\nabla^2+\kappa^4)\, f = -\frac{\delta(x-x_i\,,\,y-y_i)}{D} \tag{5.32}$$

Substituting from equation (5.31) into (5.26), then it can be deduced that:

$$\begin{bmatrix} \theta_x^* \\ \theta_y^* \\ w^* \end{bmatrix} = \underset{\sim}{C}\, f \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \underset{\sim}{U} \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} \tag{5.33}$$

where

$$\underset{\sim}{U} = \underset{\sim}{C} f = \frac{1}{\lambda^2}(\mathcal{D}^*)^t f \tag{5.34}$$

and from the definition of the cofactor matrix, it can be shown that:

$$\underset{\sim}{C} = \begin{bmatrix} \nabla^2-\lambda^2+\alpha\frac{\partial^2}{\partial y^2}(\nabla^2-\beta)+\gamma\frac{\partial^2}{\partial y^2} & -\alpha\frac{\partial^2}{\partial x \partial y}(\nabla^2+\frac{\gamma}{\alpha}-\beta) & \frac{\partial}{\partial x}(\nabla^2-\lambda^2) \\ -\alpha\frac{\partial^2}{\partial x \partial y}(\nabla^2+\frac{\gamma}{\alpha}-\beta) & \left[(\nabla^2-\lambda^2+\alpha\frac{\partial^2}{\partial x^2})(\nabla^2-\beta)+\gamma\frac{\partial^2}{\partial x^2}\right] & \frac{\partial}{\partial y}(\nabla^2-\lambda^2) \\ -\frac{\gamma}{\lambda^2}\frac{\partial}{\partial x}(\nabla^2-\lambda^2) & -\frac{\gamma}{\lambda^2}\frac{\partial}{\partial y}(\nabla^2-\lambda^2) & (\nabla^2-\lambda^2)(\frac{1+\alpha}{\lambda^2}\nabla^2-1) \end{bmatrix} \tag{5.35}$$

The fundamental solution problem reduces to the solution of differential equation (5.32) in terms of f, then other parameters are obtained by means of equation (5.34) through direct differentiation.

5.3.2 Solution of fundamental solution problem

$$(\nabla^2-\lambda^2)(\nabla^4-2b\nabla^2+\kappa^4)f = -\frac{\delta(x-x_i\ ,\ y-y_i)}{D}$$

Applying 2D Fourier transform, as defined by equation (4.60), and using the results given by equation (4.64) to (4.66), then equation (5.32) will be transformed to:

$$(-\rho^2-\lambda^2)\left\{(-\rho^2)^2-2b(-\rho^2)+\kappa^4\right\}\bar{f} = -\frac{1}{2\pi D} \tag{5.36}$$

i.e.

$$\bar{f}(\rho)=\frac{1}{2\pi D(\rho^2+\lambda^2)(\rho^4+2b\rho^2+\kappa^4)} \tag{5.37}$$

Defining λ_1^2, λ_2^2 such that

$$\lambda_1^2+\lambda_2^2=2b$$

$$\lambda_1^2\lambda_2^2=\kappa^4$$

i.e.

$$\rho^4+2b\rho^2+\kappa^4= (\rho^2+\lambda_1^2)(\rho^2+\lambda_2^2)$$

then equation (5.37) may be re-written as follows:

$$\bar{f}(\rho)=\frac{1}{2\pi D(\rho^2+\lambda^2)(\rho^2+\lambda_1^2)(\rho^2+\lambda_2^2)} \tag{5.38}$$

Several cases will result depending on λ, λ_1^2 and λ_2^2. For the sake of generality, we shall start by discussing all possible cases.

a) Cases of real λ, λ_1, λ_2

For $b^2 \geq \kappa^4$, λ,λ_1, and λ_2 are real since

$$\lambda_1^2 = b+\sqrt{b^2-\kappa^4} \tag{5.39a}$$

$$\lambda_2^2 = b-\sqrt{b^2-\kappa^4} \tag{5.39b}$$

However, options may occur as follows:

i) $\lambda=\lambda_1=\lambda_2$

i.e.

$$\bar{f}(\rho)=\frac{1}{2\pi D(\rho^2+\lambda^2)^3} \tag{5.40}$$

From [Ref 136]. If

$$\bar{U}(\rho)=\frac{1}{(\rho^2+a^2)^n}; \qquad n>\tfrac{1}{4};\ a>0 \tag{5.41a}$$

then

$$U(r)=\left(\frac{r}{2a}\right)^{n-1}\frac{K_{n-1}(ar)}{\Gamma(n)} \tag{5.41b}$$

Hence it can be deduced that:

$$f(r) = \frac{1}{2\pi D}\left(\frac{r}{2\lambda}\right)^2\frac{K_2(\lambda r)}{\Gamma(3)}$$

$$= \frac{r^2K_2(\lambda r)}{16\pi\lambda^2 D} \tag{5.42}$$

(ii) Two equal λ's

Let $\lambda_3\equiv\lambda$, and write the following expression:

$$\bar{f}(\rho)=\frac{1}{2\pi D(\rho^2+\lambda_i^2)(\rho^2+\lambda_j^2)^2} \tag{5.43}$$

where

if $\lambda_i = \lambda$ then $\lambda_j = \lambda_1 = \lambda_2$

and

if $\lambda_i = \lambda_1$ then $\lambda_j = \lambda = \lambda_2$, etc

using partial fractions, it can be shown that

$$\bar{f}(\rho) = \frac{1}{2\pi D(\lambda_i^2-\lambda_j^2)^2}\left\{\frac{1}{\rho^2+\lambda_i^2} - \frac{1}{\rho^2+\lambda_j^2}\right\} + \frac{1}{2\pi D(\lambda_i^2-\lambda_j^2)}\,\frac{1}{(\rho^2+\lambda_j^2)^2} \tag{5.44}$$

Using equation (5.41), it can be proved that:

$$f(r) = \frac{1}{2\pi D(\lambda_i^2-\lambda_j^2)^2}\left\{K_o(\lambda_i r) - K_o(\lambda_j r)\right\} + \frac{rK_1(\lambda_j r)}{4\pi D\lambda_j^2(\lambda_i^2-\lambda_j^2)} \tag{5.45}$$

(iii) $\underline{\lambda \neq \lambda_1 \neq \lambda_2}$

Let $\lambda_3 \equiv \lambda$ and using partial fractions, it can be shown that:

$$\bar{f}(\rho) = \frac{1}{2\pi D(\rho^2+\lambda_1^2)(\rho^2+\lambda_2^2)(\rho^2+\lambda_3^2)}$$

$$= \frac{\alpha_1}{\rho^2+\lambda_1^2} + \frac{\alpha_2}{\rho^2+\lambda_2^2} + \frac{\alpha_3}{\rho^2+\lambda_3^2} \tag{5.46}$$

where

$$\alpha_1 = \frac{1}{2\pi D(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)} \tag{5.47a}$$

$$\alpha_2 = \frac{1}{2\pi D(\lambda_2^2-\lambda_3^2)(\lambda_2^2-\lambda_1^2)} \tag{5.47b}$$

$$\alpha_3 = \frac{1}{2\pi D(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)} \tag{5.47c}$$

Hence, by using equation (5.41), it can be proved that:

$$f(r) = \alpha_1 K_o(\lambda_1 r) + \alpha_2 K_o(\lambda_2 r) + \alpha_3 K_o(\lambda_3 r) \tag{5.48}$$

b) Case with b=0

For such a case,

$$\bar{f}(\rho) = \frac{1}{2\pi D(\rho^2+\lambda^2)(\rho^4+\kappa^4)}$$

i.e.

$$\lambda_1^2 = -i\kappa^2,\ \lambda_2^2 = i\kappa^2,\quad i = \sqrt{-1}$$

and from previous analysis, it can be shown that:

$$\bar{f}(\rho) = \bar{f}_1(\rho) + \bar{f}_2(\rho) \tag{5.49}$$

where

$$\bar{f}_1(\rho)=\frac{1}{2\pi D(\lambda^4+\kappa^4)(\rho^2+\lambda^2)}$$

$$\bar{f}_2(\rho)=\frac{1}{4\pi D i\kappa^2}\left\{\frac{1}{(\lambda^2+i\kappa^2)}\,\frac{1}{(\rho^2-i\kappa^2)}-\frac{1}{(\lambda^2-i\kappa^2)}\,\frac{1}{(\rho^2+i\kappa^2)}\right\}$$

Hence, it can be deduced that:

$$f_1(r)=\frac{1}{2\pi D(\lambda^4+\kappa^4)}\,K_o(\lambda r) \tag{5.50}$$

and

$$f_2(r)=\frac{1}{4\pi D i\kappa^2(\lambda^4+\kappa^4)}\left\{\lambda^2\Big(K_o(\kappa r\sqrt{-i})-K_o(\kappa r\sqrt{i})\Big)-i\kappa^2\Big(K_o(\kappa r\sqrt{-i})+K_o(\kappa r\sqrt{i})\Big)\right\}$$

and from the properties of Kelvin functions, [Ref 138], it can be shown that:

$$K_o(z\sqrt{-i})-K_o(z\sqrt{i})=-2i\;K_{ei}(z)$$

$$K_o(z\sqrt{-i})+K_o(z\sqrt{i})=2\;K_{er}(z)$$

i.e.

$$f_2(r)\;=\;-\;\frac{1}{2\pi D(\lambda^4-\kappa^4)}\left\{\frac{\lambda^2}{\kappa^2}K_{ei}(\kappa r)+K_{er}(\kappa r)\right\} \tag{5.51}$$

and f(r) can, therefore, be expressed as follows:

$$f(r)\;=\;\frac{1}{2\pi D(\lambda^4-\kappa^4)}\left\{K_o(\lambda r)-\frac{\lambda^2}{\kappa^2}K_{ei}(\kappa r)+K_{er}(\kappa r)\right\} \tag{5.52}$$

c) <u>Case with $b^2<\kappa^4$</u>

Using equation (5.46), (5.47) with

$$\lambda_1^2=b+\left(\sqrt{\kappa^4-b^2}\right)i$$

$$\lambda_2^2=b-\left(\sqrt{\kappa^4-b^2}\right)i$$

writing $\lambda_3\equiv\lambda$, then it can be deduced from equation (5.48) that:

$$f(r)=\alpha_1K_o(\lambda_1r)+\alpha_2K_o(\lambda_2r)+\alpha_3K_o(\lambda_3r) \tag{5.53}$$

where α_1, α_2 and α_3 are as defined by (5.47). Notice that if f(r) is divided into two terms such that:

$$f(r)=f_{12}(r)+f_3(r) \tag{5.54}$$

where

$f_3(r)=\alpha_3K_o(\lambda_3r)\equiv\alpha_3K_o(\lambda r)$, then it can be shown that:

$$f_{12}(r)=\frac{1}{2\pi D(\lambda^4-2b\lambda^2+\kappa^4)}\left\{\frac{(\lambda^2-\lambda_2^2)K_o(\lambda_1r)-(\lambda^2-\lambda_1^2)K_o(\lambda_2r)}{(\lambda_1^2-\lambda_2^2)}\right\}$$

and from the definition of λ_1 and λ_2, it can shown that:

$$f_{12}(r)=-\frac{1}{2\pi D(\lambda^4-2b\lambda^2+\kappa^4)(2i)\sqrt{\kappa^4-b^2}}\left\{(\lambda^2-b)\Big(K_o(\lambda_1r)-K_o(\lambda_2r)\Big)\right.$$

$$\left.+\sqrt{\kappa^4-b^2}\,i\,\Big(K_o(\lambda_1r)+K_o(\lambda_2r)\Big)\right\} \tag{5.55}$$

Using exponential representation for λ_1 and λ_2, i.e.

$$\lambda_1^2=\kappa^2e^{2i\phi} \tag{5.56a}$$

$$\lambda_2^2=\kappa^2e^{-2i\phi} \tag{5.56b}$$

where

$$\tan(2\phi)=\frac{\sqrt{\kappa^4-b^2}}{b} \tag{5.57}$$

then

$$\lambda_1 = \kappa e^{i\phi}$$

$$\lambda_2 = \kappa e^{-i\phi}$$

Let $z=\lambda_1 r$, $\bar{z}=\lambda_2 r$, then "$\bar{z}$" is the complex conjugate of z. Hence:

$$f_{12}(r) = -\frac{1}{2\pi D(\lambda^4-2b\lambda^2+\kappa^4)(2i)\sqrt{\kappa^4-b^2}}\Big\{(\lambda^2-b)\big(K_o(z)-K_o(\bar{z})\big)$$

$$+\left(\sqrt{\kappa^4-b^2}\right)i\ \big(K_o(z)+K_o(\bar{z})\big)\Big\} \tag{5.58}$$

From the properties of Bessel functions, [Ref 138], it can be deduced that:

$$K_o(z) - K_o(\bar{z}) = 2i\ \mathfrak{I}_m\big(K_o(z)\big)$$

$$K_o(z) + K_o(\bar{z}) = 2\ \mathfrak{R}_e\big(K_o(z)\big)$$

Hence, it can be proved that:

$$f_{12}(r) = -\frac{1}{2\pi D(\lambda^4-2b\lambda^2+\kappa^4)}\left\{\frac{(\lambda^2-b)}{\sqrt{\kappa^4-b^2}}\mathfrak{I}_m\big(K_o(z)\big)+\mathfrak{R}_e\big(K_o(z)\big)\right\} \tag{5.59}$$

where

$$z=\kappa r e^{i\phi} = \kappa r\big(\cos(\phi)+i\sin(\phi)\big) \tag{5.60}$$

and the total fundamental solution can, therefore be expressed as follows:

$$f_{12}(r) = \frac{1}{2\pi D(\lambda^4-2b\lambda^2+\kappa^4)}\left\{K_o(\lambda r)-\mathfrak{R}_e\big(K_o(z)\big)-\frac{(\lambda^2-b)}{\sqrt{\kappa^4-b^2}}\mathfrak{I}_m\big(K_o(z)\big)\right\} \tag{5.61}$$

which proves that such a fundamental solution is real. Throughout the analysis and computer programs, equation (5.53) is used for the fundamental solution since it is easier to differentiate than equation (5.61), and the real parts of the resulting expressions will only be considered.

5.4 BOUNDARY INTEGRAL EQUATIONS

Substituting from equation (5.23) into (5.21) and using the following property of the Dirac-delta function:

$$\iint_{\Omega} f(x,y)\delta(x-x_i,y-y_i)\, dx\, dy = C_i\, f(x_i,y_i) \tag{5.62}$$

where

$$C_i = 0 \qquad \text{if } (x_i,y_i) \text{ is outside } \Omega$$

$$= \tfrac{1}{2} \qquad \text{if } (x_i,y_i) \text{ is on } \Gamma$$

$$= 1 \qquad \text{if } (x_i,y_i) \text{ is inside } \Omega,$$

then, it can be deduced that:

$$C_i\Big\{w_i+(\theta_x)_i+(\theta_y)_i\Big\}+ \oint_{\Gamma}\Big(\theta_x\, t_x^* + \theta_y\, t_y^* + w\, t_z^*\Big)\, ds$$
$$=\oint_{\Gamma}\Big(\theta_x^*\, t_x + \theta_y^*\, t_y + w^* t_z\Big)\, ds+ \iint_{\Omega} q\Big(w^* - \frac{\nu}{(1-\nu)\lambda^2}\big(\frac{\partial\theta_x^*}{\partial x} + \frac{\partial\theta_y^*}{\partial y}\big)\Big)\, dx\, dy \tag{5.63}$$

It may be useful for boundary conditions to use normal and tangential components, defined as follows:

$$\begin{bmatrix} \theta_n^* \\ \theta_t^* \\ w^* \end{bmatrix} = \begin{bmatrix} l & m & 0 \\ -m & l & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \theta_x^* \\ \theta_y^* \\ w^* \end{bmatrix} \tag{5.64}$$

and defining ϕ^* such that:

$$\phi^* = w^* - \frac{\nu}{(1-\nu)\lambda^2}\big(\frac{\partial\theta_x^*}{\partial x} + \frac{\partial\theta_y^*}{\partial y}\big)$$

$$\equiv\phi_1 e_x+\phi_2 e_y+\phi_3 e_z$$

Hence, equation (5.63) may be reduced as follows:

$$C_i\left\{w_i+(\theta_x)_i+(\theta_y)_i\right\}+\oint_\Gamma\left(M^*_n\theta_n+M^*_{tn}\,\theta_t+Q^*_n\,w\right)ds$$

$$=\oint_\Gamma\left(M_n\theta^*_n+M_{tn}\theta^*_t+Q_n\,w^*\right)ds+\iint_\Omega q\phi^*\,dx\,dy \qquad (5.65)$$

If the normal and tangential axes are used instead of x–y axes, it can be proved that:

$$M^*_n = D\left(\frac{\partial\theta^*_n}{\partial n}+\nu\frac{\partial\theta^*_t}{\partial t}\right) \qquad (5.66a)$$

$$M^*_{tn} = D\,\frac{(1-\nu)}{2}\left(\frac{\partial\theta^*_n}{\partial t}+\frac{\partial\theta^*_t}{\partial n}\right) \qquad (5.66b)$$

$$Q^*_n = (1-\nu)\frac{D}{2}\lambda^2\left(\theta^*_n+\frac{\partial w^*}{\partial n}\right) \qquad (5.66c)$$

defining $\underline{g}$, $\underline{h}$ matrices such that:

$$\begin{bmatrix}\theta^*_n\\ \theta^*_t\\ w^*\end{bmatrix}=\underline{g}\begin{bmatrix}e_x\\ e_y\\ e_z\end{bmatrix} \qquad (5.67)$$

and

$$\begin{bmatrix}M^*_n\\ M^*_{tn}\\ Q^*_n\end{bmatrix}=\underline{h}\begin{bmatrix}e_x\\ e_y\\ e_z\end{bmatrix} \qquad (5.68)$$

then it can be proved from equations (5.33), (5.64) and (5.67), (5.68) that:

$$\underline{g}=\begin{bmatrix}l & m & 0\\ -m & l & 0\\ 0 & 0 & 1\end{bmatrix}\underline{U} \qquad (5.69)$$

$$\underline{h}=D\begin{bmatrix}\frac{\partial}{\partial n} & \nu\frac{\partial}{\partial t} & 0\\ \frac{(1-\nu)}{2}\frac{\partial}{\partial t} & \frac{(1-\nu)}{2}\frac{\partial}{\partial n} & 0\\ \frac{(1-\nu)}{2}\lambda^2 & 0 & \frac{(1-\nu)}{2}\lambda^2\frac{\partial}{\partial n}\end{bmatrix}\underline{g} \qquad (5.70)$$

Substituting from equation (5.67), (5.68) into (5.65) and considering arbitrary e_x, e_y, e_z, the following boundary integral equations are obtained:

$$C_i\,(\theta_x)_i + \oint_\Gamma \left(h_{11}\theta_n + h_{21}\theta_t + h_{31}w\right) ds$$
$$= \oint_\Gamma \left(g_{11}M_n + g_{21}M_{tn} + g_{31}Q_n\right) ds + \iint_\Omega q\phi_1 \, dx\, dy \qquad (5.71a)$$

$$C_i\,(\theta_y)_i + \oint_\Gamma \left(h_{12}\theta_n + h_{22}\theta_t + h_{32}w\right) ds$$
$$= \oint_\Gamma \left(g_{12}M_n + g_{22}M_{tn} + g_{32}Q_n\right) ds + \iint_\Omega q\phi_2 \, dx\, dy \qquad (5.71b)$$

$$C_i\, w_i + \oint_\Gamma \left(h_{13}\theta_n + h_{23}\theta_t + h_{33}w\right) ds$$
$$= \oint_\Gamma \left(g_{13}M_n + g_{23}M_{tn} + g_{33}Q_n\right) ds + \iint_\Omega q\phi_3 \, dx\, dy \qquad (5.71c)$$

5.5 FUNDAMENTAL SOLUTION PARAMETERS FOR PRACTICAL ENGINEERING CASES

From a practical point of view, it is clear that for thin plates and moderately thick plates and for foundations, used in engineering applications, it is expected that

$$b^2 < \kappa^4$$

i.e.

$$\lambda \neq \lambda_1 \neq \lambda_2,$$

$$\lambda_1^2 = b + \left(\sqrt{\kappa^4 - b^2}\right) i$$

$$\lambda_2^2 = b - \left(\sqrt{\kappa^4 - b^2}\right) i$$

Hence, fundamental solution parameters and computer programs will be based on fundamental solutions defined by equation (5.53). Using equation (5.33), it can be deduced that:

$$\underline{U} = \alpha_1\bar{\underline{U}}(\lambda_1) + \alpha_2\bar{\underline{U}}(\lambda_2) + \alpha_3\bar{\underline{U}}(\lambda_3) \tag{5.72}$$

where

$$\bar{\underline{U}}(\lambda_s) = \underline{c}K_o(\lambda_s r) \tag{5.73}$$

Using equations (5.69), (5.70) and (5.72), it can be shown that:

$$\underline{g} = \alpha_1\bar{\underline{g}}(\lambda_1) + \alpha_2\bar{\underline{g}}(\lambda_2) + \alpha_3\bar{\underline{g}}(\lambda_3) \tag{5.74}$$

and

$$\underline{h} = \alpha_1\bar{\underline{h}}(\lambda_1) + \alpha_2\bar{\underline{h}}(\lambda_2) + \alpha_3\bar{\underline{h}}(\lambda_3) \tag{5.75}$$

where

$$\bar{\underline{g}}(\lambda_s) = \begin{bmatrix} l & m & 0 \\ -m & l & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{\underline{U}}(\lambda_s) \tag{5.76}$$

$$\bar{\underline{h}}(\lambda_s) = D \begin{bmatrix} \frac{\partial}{\partial n} & \nu\frac{\partial}{\partial t} & 0 \\ \frac{(1-\nu)}{2}\frac{\partial}{\partial t} & \frac{(1-\nu)}{2}\frac{\partial}{\partial n} & 0 \\ \frac{(1-\nu)}{2}\lambda^2 & 0 & \frac{(1-\nu)}{2}\lambda^2\frac{\partial}{\partial n} \end{bmatrix} \bar{\underline{g}}(\lambda_s) \tag{5.77}$$

From the properties of Bessel functions, Appendix C, it can be deduced that

$$\frac{\partial}{\partial\xi} K_o(cr) = -c\, K_1(cr)\frac{\partial r}{\partial\xi} \tag{5.78a}$$

$$\frac{\partial}{\partial \xi} K_1(cr) = -\frac{\partial r}{\partial \xi}\left(c\ K_o(cr)+\frac{1}{r}K_1(cr)\right) \tag{5.78b}$$

$$\nabla^2 K_o(cr)=c^2 K_o(cr) \tag{5.78c}$$

Hence, it can be proved that:

$$\bar{\underline{U}}(c)=\underline{A}(c)K_o(cr)+\underline{B}(c)K_1(cr) \tag{5.79}$$

where $\underline{A}(c)$, $\underline{B}(c)$ are as listed in tables 5.1 and 5.2 respectively.

$$A_{11}(c)=\left\{c^2-\lambda^2+\alpha c^2\left(\frac{\partial r}{\partial y}\right)^2\right\}(c^2-\beta)+\gamma c^2\left(\frac{\partial r}{\partial y}\right)^2$$

$$A_{12}(c)=-\alpha c^2\left(c^2+\frac{\gamma}{\alpha}-\beta\right)\frac{\partial r}{\partial x}\cdot\frac{\partial r}{\partial y}$$

$$A_{13}(c)=0$$

$$A_{21}(c)=-\alpha c^2\left(c^2+\frac{\gamma}{\alpha}-\beta\right)\frac{\partial r}{\partial x}\cdot\frac{\partial r}{\partial y}$$

$$A_{22}(c)=\left\{c^2-\lambda^2+\alpha c^2\left(\frac{\partial r}{\partial x}\right)^2\right\}(c^2-\beta)+\gamma c^2\left(\frac{\partial r}{\partial x}\right)^2$$

$$A_{23}(c)=0$$

$$A_{31}(c)=0$$

$$A_{32}(c)=0$$

$$A_{33}(c)=(c^2-\lambda^2)\left\{\frac{(\alpha+1)c^2}{\lambda^2}-1\right\}$$

where

$$\lambda^2 = \frac{10}{h^2}, \quad \alpha = \frac{(1+\nu)}{(1-\nu)}, \quad \beta = \frac{2K}{D(1-\nu)\lambda^2}, \quad \gamma = \lambda^2 + \frac{\nu}{(1-\nu)}\beta$$

[Table 5.1] A_{ij} parameters

$$B_{11}(c)=\left\{\frac{\alpha c}{r}\left(2(\frac{\partial r}{\partial y})^2-1\right)\right\}(c^2-\beta)+\frac{\gamma c}{r}\left(2(\frac{\partial r}{\partial y})^2-1\right)$$

$$B_{12}(c)=-\frac{2\alpha c}{r}(c^2+\frac{\gamma}{\alpha}-\beta)\frac{\partial r}{\partial x}\cdot\frac{\partial r}{\partial y}$$

$$B_{13}(c)=-c(c^2-\lambda^2)\frac{\partial r}{\partial x}$$

$$B_{21}(c)=-\frac{2\alpha c}{r}(c^2+\frac{\gamma}{\alpha}-\beta)\frac{\partial r}{\partial x}\cdot\frac{\partial r}{\partial y}$$

$$B_{22}(c)=\left\{\frac{\alpha c}{r}\left(2(\frac{\partial r}{\partial x})^2-1\right)\right\}(c^2-\beta)+\frac{\gamma c}{r}\left(2(\frac{\partial r}{\partial x})^2-1\right)$$

$$B_{23}(c)=-c(c^2-\lambda^2)\frac{\partial r}{\partial y}$$

$$B_{31}(c)=\frac{c\gamma}{\lambda^2}(c^2-\lambda^2)\frac{\partial r}{\partial x}$$

$$B_{32}(c)=\frac{c\gamma}{\lambda^2}(c^2-\lambda^2)\frac{\partial r}{\partial y}$$

$$B_{33}(c)=0$$

[Table 5.2] B_{ij} parameters

Using equations (5.76) and (5.79), it can be shown that:

$$\underline{g}(c)=\underline{A}'(c)K_o(cr)+\underline{B}'(c)K_1(cr) \tag{5.80}$$

where

$$\underline{A}'(c) = \begin{bmatrix} l & m & 0 \\ -m & l & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{A}(c) \tag{5.81}$$

$$\underline{B}'(c) = \begin{bmatrix} l & m & 0 \\ -m & l & 0 \\ 0 & 0 & 1 \end{bmatrix} \underline{B}(c) \tag{5.82}$$

Similarly, from equation (5.77), (5.80), it can be proved that:

$$\bar{\underline{h}}(c)=\underline{E}(c)K_o(cr)+\underline{F}(c)K_1(cr) \tag{5.83}$$

The full derivation of matrices $\underline{E}(c)$ and $\underline{F}(c)$ is given in Appendix F.

5.6 LOADING TERMS

5.6.1 Concentrated loading

Consider for simplicity a concentrated load Q_z acting at a point (x_o,y_o), then equivalent "q" may be expressed as follows:

$$q=Q_z\delta(x-x_o,y-y_o) \tag{5.84}$$

Hence, it can be deduced that:

$$\iint q\phi^*(x-x_i,y-y_i)\ dx\ dy = q\phi^*(x_o-x_i,y_o-y_i) \tag{5.85}$$

where

$$\phi^* = w^* - \frac{\nu}{(1-\nu)\lambda^2} \left(\frac{\partial\theta_x^*}{\partial x} + \frac{\partial\theta_y^*}{\partial y} \right)$$

From equations (5.33), (5.72), it can be deduced that:

$$\phi^{*}=p_1^{*}e_x+p_2^{*}e_y+p_3^{*}e_z \tag{5.86}$$

where

$$p_s^{*}=\alpha_1\bar{p}_s(\lambda_1)+\alpha_2\bar{p}_s(\lambda_2)+\alpha_3\bar{p}_s(\lambda_3)$$
$$s=1,\ 2,\ 3.$$

and

$$\bar{p}_1(c)=\bar{U}_{31}(c)-\frac{\nu}{(1-\nu)\lambda^2}\left(\frac{\partial\bar{U}_{11}}{\partial x}+\frac{\partial\bar{U}_{21}}{\partial y}\right) \tag{5.87a}$$

$$\bar{p}_2(c)=\bar{U}_{32}(c)-\frac{\nu}{(1-\nu)\lambda^2}\left(\frac{\partial\bar{U}_{12}}{\partial x}+\frac{\partial\bar{U}_{22}}{\partial y}\right) \tag{5.87b}$$

$$\bar{p}_3(c)=\bar{U}_{33}(c)-\frac{\nu}{(1-\nu)\lambda^2}\left(\frac{\partial\bar{U}_{13}}{\partial x}+\frac{\partial\bar{U}_{23}}{\partial y}\right) \tag{5.87c}$$

and from

$$\bar{U}_{ij}(c)=A_{ij}(c)K_o(cr)+B_{ij}(c)K_1(cr)$$

it can be proved that:

$$\bar{p}_1(c)=\bar{U}_{31}(c)-\frac{\nu}{(1-\nu)\lambda^2}\Big\{\Big(\Big(\frac{\partial A_{11}}{\partial x}+\frac{\partial A_{21}}{\partial y}\Big)-c\Big(B_{11}\frac{\partial r}{\partial x}+B_{21}\frac{\partial r}{\partial y}\Big)\Big)K_o(cr)$$
$$+\Big(\Big(\frac{\partial B_{11}}{\partial x}+\frac{\partial B_{21}}{\partial y}\Big)-\Big(cA_{11}+\frac{B_{11}}{r}\Big)\frac{\partial r}{\partial x}-\Big(cA_{21}+\frac{B_{21}}{r}\Big)\frac{\partial r}{\partial y}\Big)K_1(cr)\Big\} \tag{5.88}$$

$$\bar{p}_2(c)=\bar{U}_{32}(c)-\frac{\nu}{(1-\nu)\lambda^2}\Big\{\Big(\Big(\frac{\partial A_{12}}{\partial x}+\frac{\partial A_{22}}{\partial y}\Big)-c\Big(B_{12}\frac{\partial r}{\partial x}+B_{22}\frac{\partial r}{\partial y}\Big)\Big)K_o(cr)$$
$$+\Big(\Big(\frac{\partial B_{12}}{\partial x}+\frac{\partial B_{22}}{\partial y}\Big)-\Big(cA_{12}+\frac{B_{12}}{r}\Big)\frac{\partial r}{\partial x}-\Big(cA_{22}+\frac{B_{22}}{r}\Big)\frac{\partial r}{\partial y}\Big)K_1(cr)\Big\} \tag{5.89}$$

$$\bar{p}_3(c)=\bar{U}_{33}(c)-\frac{\nu}{(1-\nu)\lambda^2}\Big\{\Big(-c\Big(B_{13}\frac{\partial r}{\partial x}+B_{23}\frac{\partial r}{\partial y}\Big)\Big)K_o(cr)$$
$$+\Big(\Big(\frac{\partial B_{13}}{\partial x}+\frac{\partial B_{23}}{\partial y}\Big)-\frac{B_{13}}{r}\frac{\partial r}{\partial x}+\frac{B_{23}}{r}\frac{\partial r}{\partial y}\Big)K_1(cr)\Big\} \tag{5.90}$$

and the derivatives of A_{ij}, B_{ij} are as listed in table 5.3

$$\frac{\partial A_{11}(c)}{\partial x}=-\frac{2c^2}{r}\left\{\alpha(c^2-\beta)+\gamma\right\}\frac{\partial r}{\partial x}\cdot\left(\frac{\partial r}{\partial y}\right)^2$$

$$\frac{\partial A_{12}(c)}{\partial x}=-\frac{\alpha c^2}{r}\left(c^2+\frac{\gamma}{\alpha}-\beta\right)\left\{\frac{\partial r}{\partial y}-2\left(\frac{\partial r}{\partial x}\right)^2\frac{\partial r}{\partial y}\right\}$$

$$\frac{\partial A_{13}(c)}{\partial x}=0$$

$$\frac{\partial A_{21}(c)}{\partial y}=-\frac{\alpha c^2}{r}\left(c^2+\frac{\gamma}{\alpha}-\beta\right)\left\{\frac{\partial r}{\partial x}-2\frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial y}\right)^2\right\}$$

$$\frac{\partial A_{22}(c)}{\partial y}=-\frac{2c^2}{r}\left\{\alpha(c^2-\beta)+\gamma\right\}\left(\frac{\partial r}{\partial x}\right)^2\cdot\frac{\partial r}{\partial y}$$

$$\frac{\partial A_{23}(c)}{\partial y}=0$$

$$\frac{\partial B_{11}(c)}{\partial x}=\frac{c}{r^2}\left\{\alpha(c^2-\beta)+\gamma\right\}\left\{\frac{\partial r}{\partial x}-6.\frac{\partial r}{\partial x}\cdot\left(\frac{\partial r}{\partial y}\right)^2\right\}$$

$$\frac{\partial B_{12}(c)}{\partial x}=-\frac{2\alpha c}{r^2}\left(c^2+\frac{\gamma}{\alpha}-\beta\right)\left\{\frac{\partial r}{\partial y}-3.\left(\frac{\partial r}{\partial x}\right)^2\frac{\partial r}{\partial y}\right\}$$

$$\frac{\partial B_{13}(c)}{\partial x}=-\frac{c}{r}(c^2-\lambda^2)\left[1-\left(\frac{\partial r}{\partial x}\right)^2\right]$$

$$\frac{\partial B_{21}(c)}{\partial y}=-\frac{2\alpha c}{r^2}\left(c^2+\frac{\gamma}{\alpha}-\beta\right)\left\{\frac{\partial r}{\partial x}-3.\frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial y}\right)^2\right\}$$

$$\frac{\partial B_{22}(c)}{\partial y}=\frac{c}{r^2}\left\{\alpha(c^2-\beta)+\gamma\right\}\left\{\frac{\partial r}{\partial y}-6.\left(\frac{\partial r}{\partial x}\right)^2\cdot\frac{\partial r}{\partial y}\right\}$$

$$\frac{\partial B_{23}(c)}{\partial y}=-\frac{c}{r}(c^2-\lambda^2)\left(1-\left(\frac{\partial r}{\partial y}\right)^2\right)$$

[Table 5.3] Derivatives of A_{ij} and B_{ij}

5.6.2 Distributed loading

Consider for simplicity a case with constant "q",then,

$$\iint_\Omega q\phi^* \, dx\, dy = \iint_\Omega q\Big(w^* - \frac{\nu}{(1-\nu)\lambda^2}\Big(\frac{\partial\theta_x^*}{\partial x} + \frac{\partial\theta_y^*}{\partial y}\Big)\Big)\, dx\, dy$$

$$= \iint_\Omega q w^* dx\, dy - \frac{\nu q}{(1-\nu)\lambda^2}\oint_\Gamma \theta_n^* \, ds$$

and from the previous analysis,

$$\theta_n^* = g_{11}e_x + g_{12}e_y + g_{13}e_z,$$

$$w^* = \Big\{e_x[-\tfrac{\gamma}{\lambda^2}(\nabla^2-\lambda^2)]\frac{\partial}{\partial x} + e_y[-\tfrac{\gamma}{\lambda^2}(\nabla^2-\lambda^2)]\frac{\partial}{\partial y} + e_z(\nabla^2-\lambda^2)\Big[\frac{\alpha+1}{\lambda^2}\nabla^2 - 1\Big]\Big\}\frac{f^*}{D}$$

$$= \alpha_1\bar{w}(\lambda_1) + \alpha_2\bar{w}(\lambda_2) + \alpha_3\bar{w}(\lambda_3)$$

where

$$\bar{w}(c) = \Big\{e_x[-\tfrac{\gamma}{\lambda^2}(\nabla^2-\lambda^2)]\frac{\partial}{\partial x} + e_y[-\tfrac{\gamma}{\lambda^2}(\nabla^2-\lambda^2)]\frac{\partial}{\partial y} + e_z(\nabla^2-\lambda^2)[\tfrac{\alpha+1}{\lambda^2}\nabla^2-1]\Big\}K_o(cr)$$

$$= e_x[-\tfrac{\gamma}{\lambda^2}(c^2-\lambda^2)]\frac{\partial}{\partial x} + e_y[-\tfrac{\gamma}{\lambda^2}(c^2-\lambda^2)]\frac{\partial}{\partial y} + e_z\Big[(c^2-\lambda^2)\Big\{(\tfrac{\alpha+1}{\lambda^2})c^2-1\Big\}\Big]\frac{\nabla^2}{c^2}K_o(cr)$$

$$+ e_z(c^2-\lambda^2)\Big\{(\tfrac{\alpha+1}{\lambda^2})c^2-1\Big\}\,\frac{2\pi}{c^2}\delta(x-x_i;y-y_i)$$

Hence, it can be shown that:

$$\iint_\Omega q\bar{w}(c)\, dx\, dy = \oint_\Gamma q\,\Big\{\, l\, e_x[-\tfrac{\gamma}{\lambda^2}(c^2-\lambda^2)] + m\, e_y[-\tfrac{\gamma}{\lambda^2}(c^2-\lambda^2)]$$

$$+ e_z(c^2-\lambda^2)[\tfrac{\alpha+1}{\lambda^2}c^2-1]\frac{1}{c^2}\frac{\partial}{\partial n}\Big\}K_o\, ds + e_z q s$$

from which it can be proved that:

$$\iint_\Omega q\phi^*\, dx\, dy = \oint_\Gamma q\Big\{L_1^* e_x + L_2^* e_y + L_3^* e_z\Big\}ds$$

where

$$L_s^* = \alpha_1\bar{L}_s(\lambda_1) + \alpha_2\bar{L}_s(\lambda_2) + \alpha_3\bar{L}_s(\lambda_3)$$

and

$$\bar{L}_1(c)=-\frac{\gamma\, l}{\lambda^2}(c^2-\lambda^2)K_o(cr)-\frac{\nu}{(1-\nu)\lambda^2}\bar{g}_{11}(c)$$

$$\bar{L}_2(c)=-\frac{\gamma\, m}{\lambda^2}(c^2-\lambda^2)K_o(cr)-\frac{\nu}{(1-\nu)\lambda^2}\bar{g}_{12}(c)$$

$$\bar{L}_3(c)=-(c^2-\lambda^2)[\frac{\alpha+1}{\lambda^2}c^2-1]\frac{1}{c}\frac{\partial r}{\partial n}K_1(cr)-\frac{\nu}{(1-\nu)\lambda^2}\bar{g}_{13}(c)$$

where

$$s=\frac{2\pi}{c^2}(c^2-\lambda^2)\left\{(\frac{\alpha+1}{\lambda^2})c^2-1\right\}$$

5.6.3 Boundary integral equations with loading terms

Equation (5.71) can be modified as follows:

$$C_i(\theta_x)_i+\oint_\Gamma(h_{11}\theta_n+h_{21}\theta_t+h_{31}w)\;ds=\oint_\Gamma(g_{11}M_n+g_{21}M_{tn}+g_{31}Q_n)\;ds$$
$$+\oint_\Gamma q\;L_1^*\;ds+\;Q_z p_1^*(x_o-x_i,y_o-y_i)$$

$$C_i(\theta_y)_i+\oint_\Gamma(h_{12}\theta_n+h_{22}\theta_t+h_{32}w)\;ds=\oint_\Gamma(g_{12}M_n+g_{22}M_{tn}+g_{32}Q_n)\;ds$$
$$+\oint_\Gamma q\;L_2^*\;ds+\;Q_z p_2^*(x_o-x_i,y_o-y_i)$$

$$C_i w_i+\oint_\Gamma(h_{13}\theta_n+h_{23}\theta_t+h_{33}w)\;ds=\oint_\Gamma(g_{13}M_n+g_{23}M_{tn}+g_{33}Q_n)\;ds$$
$$+\oint_\Gamma q\;L_3^*\;ds+\;Q_z p_3^*(x_o-x_i,y_o-y_i)$$

where

(x_i,y_i) represents a source point,

(x_o,y_o) is the point at which Q_z is acting.

5.7 SINGULARITY CONSIDERATIONS

It is clear from the definition of $K_o(cr)$, $K_1(cr)$ and their derivatives that singular terms of order log r, $\frac{1}{r}$ and $\frac{1}{r^2}$ are developed in the fundamental solution parameters presented in section 5.5 and 5.6, and some of those parameters may lead to divergent integrals.

For the case of thick plates, it is not possible to separate the effect of foundation since it is coupled with the effect of thickness and an analysis similar to that discussed in section (4.3) is not possible.

An alternative solution is possible using the idea of "Regular Boundary Elements", introduced for 2D and 3D elasticity problems by El-Sebai [Ref 136]. In this approach, the source points, for the solution of boundary parameters, are not taken on the actual boundary itself, but they are selected on a fictitious boundary outside the domain and parallel to the actual boundary as shown in Figure 5.1. Hence, for boundary values,

$$C_i=0$$

$$r=\sqrt{(x-x_i)^2+(y-y_i)^2}>0$$

always.

The optimum distance between the fictitious boundary and actual boundary is within the order of element length. [Ref 136]. This method can also be employed for thin plates and there will be no singular terms resulting in the boundary integral equations.

5.8 ANALYSIS USING THIN-PLATE FUNDAMENTAL SOLUTION

An alternative solution based upon the fundamental solution of thin-plates on elastic foundation is presented in this section:

Consider the basic governing equations, as given by equation (4.16). For the case of an approximate solution which satisfies the boundary conditions, a weighted-residual expression can be written as follows:

$$\iint_\Omega \left\{ \theta_x^*\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x\right) + \theta_y^*\left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y\right) \right\} dx\, dy$$
$$+\iint_\Omega w^* \left\{ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q - Kw \right\} dx\, dy = 0 \tag{5.91}$$

Using integration by parts theorem, then equation (5.91) may be rewritten as follows:

$$\oint_\Gamma \left(\theta_x^*\, t_x + \theta_y^*\, t_y + w^* t_z\right) ds - \iint_\Omega \left\{ Q_x\left(\frac{\partial w^*}{\partial x} + \theta_x^*\right) + Q_y\left(\frac{\partial w^*}{\partial y} + \theta_y^*\right) \right\} dx\, dy$$

$$+\iint_\Omega w^*(q-Kw)\, dx\, dy - \iint_\Omega \left\{ M_x \frac{\partial \theta_x^*}{\partial x} + M_{xy}\left(\frac{\partial \theta_x^*}{\partial y} + \frac{\partial \theta_y^*}{\partial x}\right) + M_y \frac{\partial \theta_y^*}{\partial y} \right\} dx\, dy = 0 \tag{5.92}$$

For a moderately thick plate, the following assumptions will be made so as to simplify the previous expression:

$$\theta_x^* = -\frac{\partial w^*}{\partial x}, \qquad \theta_y^* = -\frac{\partial w^*}{\partial y} \tag{5.93}$$

and equation (5.92) may be reduced as follows:

$$\oint_\Gamma \left(\theta_x^*\, t_x + \theta_y^*\, t_y + w^* t_z\right) ds + \iint_\Omega w^*(q-Kw)\, dx\, dy$$

$$+\iint_\Omega \left\{ M_x \frac{\partial^2 w^*}{\partial x^2} + M_{xy}\left(2\frac{\partial^2 w^*}{\partial x \partial y}\right) + M_y \frac{\partial^2 w^*}{\partial y^2} \right\} dx\, dy = 0 \tag{5.94}$$

Using equation (5.11), it can be shown that:

$$M_x \frac{\partial^2 w^*}{\partial x^2} + M_y \frac{\partial^2 w^*}{\partial y^2} + 2M_{xy}\frac{\partial^2 w^*}{\partial x \partial y} = \frac{\partial \theta_x}{\partial x}\, D\left(\frac{\partial^2 w^*}{\partial x^2} + \nu \frac{\partial^2 w^*}{\partial y^2}\right) + D(1-\nu)\frac{\partial^2 w^*}{\partial x \partial y}\frac{\partial \theta_x}{\partial y}$$

$$+\frac{\partial \theta_y}{\partial y}\, D\left(\frac{\partial^2 w^*}{\partial y^2} + \nu \frac{\partial^2 w^*}{\partial x^2}\right) + D(1-\nu)\frac{\partial^2 w^*}{\partial x \partial y}\frac{\partial \theta_y}{\partial x} + \frac{\nu}{(1-\nu)\lambda^2}(q-Kw)\nabla^2 w^* \tag{5.95}$$

Defining the following fundamental solution parameters:

$$M_x^* = -D\left(\frac{\partial^2 w^*}{\partial x^2} + \nu \frac{\partial^2 w^*}{\partial y^2}\right) \tag{5.96a}$$

$$M_y^* = -D\left(\nu\,\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2}\right) \tag{5.96b}$$

$$M_{xy}^* = -\,D\,(1-\nu)D\,\frac{\partial^2 w^*}{\partial x \partial y} \tag{5.96c}$$

then equation (5.95) may be rewritten as follows:

$$M_x\frac{\partial^2 w^*}{\partial x^2}+M_y\frac{\partial^2 w^*}{\partial y^2}+2M_{xy}\frac{\partial^2 w^*}{\partial x \partial y}=-\left\{\frac{\partial \theta_x}{\partial x}M_x^*+\frac{\partial \theta_x}{\partial y}M_{xy}^*+\frac{\partial \theta_y}{\partial x}M_{xy}^*+\frac{\partial \theta_y}{\partial y}M_y^*\right\}$$

$$+\frac{\nu}{(1-\nu)\lambda^2}(q-Kw)\nabla^2 w^* \tag{5.97}$$

Hence, integrating by parts, it can be shown that

$$\iint\limits_{\Omega}\left\{M_x\frac{\partial^2 w^*}{\partial x^2}+2M_{xy}\frac{\partial^2 w^*}{\partial x \partial y}+M_y\frac{\partial^2 w^*}{\partial y^2}\right\}dx\,dy=\iint\limits_{\Omega}\frac{\nu}{(1-\nu)\lambda^2}(q-Kw)\nabla^2 w^*\,dx\,dy$$

$$-\oint\limits_{\Gamma}\left(\theta_x t_x^*+\theta_y t_y^*\right)ds+\iint\limits_{\Omega}\left\{\theta_x\left(\frac{\partial M_x^*}{\partial x}+\frac{\partial M_{xy}^*}{\partial y}\right)+\theta_y\left(\frac{\partial M_{xy}^*}{\partial x}+\frac{\partial M_y^*}{\partial y}\right)\right\}dx\,dy \tag{5.98}$$

Selecting Q_x^* and Q_y^* as 2 parameters which will be defined later, then:

$$\iint\limits_{\Omega} w\left\{\frac{\partial Q_x^*}{\partial x}+\frac{\partial Q_y^*}{\partial y}\right\}dx\,dy=\oint\limits_{\Gamma} w t_z^*\,ds-\iint\limits_{\Omega}\left\{Q_x^*\frac{\partial w}{\partial x}+Q_y^*\frac{\partial w}{\partial y}\right\}dx\,dy \tag{5.99}$$

where

$$t_x^* = l\,M_x^* + m\,M_{xy}^* \tag{5.100a}$$

$$t_y^* = l\,M_{xy}^* + m\,M_y^* \tag{5.100b}$$

$$t_z^* = l\,Q_x^* + m\,Q_y^* \tag{5.100c}$$

i.e.

$$\iint\limits_{\Omega} w\left\{\frac{\partial Q_x^*}{\partial x}+\frac{\partial Q_y^*}{\partial y}\right\}dx\,dy-\oint\limits_{\Gamma} w t_z^*\,ds+\iint\limits_{\Omega}\left\{Q_x^*\frac{\partial w}{\partial x}+Q_y^*\frac{\partial w}{\partial y}\right\}dx\,dy \tag{5.101}$$

Adding equations (5.101) to (5.98), and rearranging terms, then equation (5.98) may be modified as follows:

$$\iint_\Omega \left\{M_x\frac{\partial^2 w^*}{\partial x^2}+2M_{xy}\frac{\partial^2 w^*}{\partial x\partial y}+M_y\frac{\partial^2 w^*}{\partial y^2}\right\}dx\ dy=-\oint_\Gamma\left(\theta_x t_x^*+\theta_y t_y^*+wt_z^*\right)ds$$

$$+\iint_\Omega\left\{\theta_x\left(\frac{\partial M_x^*}{\partial x}+\frac{\partial M_{xy}^*}{\partial y}-Q_x^*\right)+\theta_y\left(\frac{\partial M_{xy}^*}{\partial x}+\frac{\partial M_y^*}{\partial y}-Q_y^*\right)+w\left(\frac{\partial Q_x^*}{\partial x}+\frac{\partial Q_y^*}{\partial y}\right)\right\}dx\ dy$$

$$+\iint_\Omega\frac{\nu}{(1-\nu)\lambda^2}(q-Kw)\nabla^2 w^*\ dx\ dy+\iint_\Omega\left\{Q_x^*\left(\theta_x+\frac{\partial w}{\partial x}\right)+Q_y^*\left(\theta_y+\frac{\partial w}{\partial y}\right)\right\}dx\ dy=0 \qquad (5.102)$$

Hence, equation (5.94) can be modified as follows:

$$\oint_\Gamma\left(\theta_x^*\ t_x+\theta_y^*\ t_y+w^* t_z\right)ds-\oint_\Gamma\left(\theta_x t_x^*+\theta_y t_y^*+wt_z^*\right)ds$$

$$+\iint_\Omega q\left\{w^*+\frac{\nu}{(1-\nu)\lambda^2}\nabla^2 w^*\right\}dx\ dy+\iint_\Omega\left\{Q_x^*\left(\theta_x+\frac{\partial w}{\partial x}\right)+Q_y^*\left(\theta_y+\frac{\partial w}{\partial y}\right)\right\}dx\ dy$$

$$+\iint_\Omega\left\{\theta_x\left(\frac{\partial M_x^*}{\partial x}+\frac{\partial M_{xy}^*}{\partial y}-Q_x^*\right)+\theta_y\left(\frac{\partial M_{xy}^*}{\partial x}+\frac{\partial M_y^*}{\partial y}-Q_y^*\right)\right.$$

$$\left.+w\left(\frac{\partial Q_x^*}{\partial x}+\frac{\partial Q_y^*}{\partial y}-K\left(w^*+\frac{\nu}{(1-\nu)\lambda^2}\nabla^2 w^*\right)\right)\right\}dx\ dy=0 \qquad (5.103)$$

In order to use thin-plate fundamental solution, equation (5.103) may be modified as follows:

$$\oint_\Gamma\left(\theta_x^*\ t_x+\theta_y^*\ t_y+w^* t_z\right)ds-\oint_\Gamma\left(\theta_x t_x^*+\theta_y t_y^*+wt_z^*\right)ds+\iint_\Omega q\left\{w^*+\frac{\nu}{(1-\nu)\lambda^2}\nabla^2 w^*\right\}dx\ dy$$

$$+\iint_\Omega\left\{Q_x^*\left(\theta_x+\frac{\partial w}{\partial x}\right)+Q_y^*\left(\theta_y+\frac{\partial w}{\partial y}\right)-\left(\frac{\nu K\nabla^2 w^*}{(1-\nu)\lambda^2}\right)w\right\}dx\ dy$$

$$+\iint_\Omega\left\{\theta_x\left(\frac{\partial M_x^*}{\partial x}+\frac{\partial M_{xy}^*}{\partial y}-Q_x^*\right)+\theta_y\left(\frac{\partial M_{xy}^*}{\partial x}+\frac{\partial M_y^*}{\partial y}-Q_y^*\right)+w\left(\frac{\partial Q_x^*}{\partial x}+\frac{\partial Q_y^*}{\partial y}-Kw^*\right)\right\}dx\ dy=0 \quad (5.104)$$

The fundamental solution for thin plates on elastic foundation is obtained from:

$$\frac{\partial M^*_x}{\partial x}+\frac{\partial M^*_{xy}}{\partial y}-Q^*_x=-e_x\delta(x-x_i,y-y_i) \tag{5.104a}$$

$$\frac{\partial M^*_{xy}}{\partial x}+\frac{\partial M^*_y}{\partial y}-Q^*_y=-e_y\delta(x-x_i,y-y_i) \tag{5.104b}$$

$$\frac{\partial Q^*_x}{\partial x}+\frac{\partial Q^*_y}{\partial y}-Kw^*=-e_z\delta(x-x_i,y-y_i) \tag{5.104c}$$

Notice, that, from equations (5.96), (5.104), it can be deduced that:

$$Q^*_x=-D\frac{\partial}{\partial x}(\nabla^2 w^*)+e_x\delta(x-x_i,y-y_i) \tag{5.105a}$$

$$Q^*_y=-D\frac{\partial}{\partial y}(\nabla^2 w^*)+e_y\delta(x-x_i,y-y_i) \tag{5.105b}$$

Hence;

$$\begin{aligned}&\iint_\Omega\left\{Q^*_x\left(\theta_x+\frac{\partial w}{\partial x}\right)+Q^*_y\left(\theta_y+\frac{\partial w}{\partial y}\right)-\left(\frac{\nu K\nabla^2 w^*}{(1-\nu)\lambda^2}\right)w\right\}\,dx\,dy\\&=e_xC_i\left(\theta_x+\frac{\partial w}{\partial x}\right)_i+e_yC_i\left(\theta_y+\frac{\partial w}{\partial y}\right)_i\\&-\iint_\Omega\left\{D\frac{\partial}{\partial x}(\nabla^2 w^*)\left(\theta_x+\frac{\partial w}{\partial x}\right)+D\frac{\partial}{\partial y}(\nabla^2 w^*)\left(\theta_y+\frac{\partial w}{\partial y}\right)+\frac{\nu K\nabla^2 w^* w}{(1-\nu)\lambda^2}\right\}\,dx\,dy\end{aligned} \tag{5.106}$$

and from equation (5.13);

$$D\left(\theta_x+\frac{\partial w}{\partial x}\right)=\frac{2Q_x}{(1-\nu)\lambda^2},$$

$$D\left(\theta_y+\frac{\partial w}{\partial y}\right)=\frac{2Q_y}{(1-\nu)\lambda^2}$$

i.e. equation (5.106) may be modified as follows:

$$\begin{aligned}&\iint_\Omega\left\{Q^*_x\left(\theta_x+\frac{\partial w}{\partial x}\right)+Q^*_y\left(\theta_y+\frac{\partial w}{\partial y}\right)-\frac{\nu K(\nabla^2 w^*)}{(1-\nu)\lambda^2}w\right\}\,dx\,dy\\&=e_xC_i\left(\theta_x+\frac{\partial w}{\partial x}\right)_i+e_yC_i\left(\theta_y+\frac{\partial w}{\partial y}\right)_i\\&+\frac{2}{(1-\nu)\lambda^2}\left\{-\oint_\Gamma Q_n\nabla^2 w^* ds+\iint_\Gamma\nabla^2 w^2\left(\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}-\frac{\nu Kw}{2}\right)\,dx\,dy\right\}\end{aligned} \tag{5.107}$$

and using $\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q-Kw=0$, the previous equation may be simplified as follows:

$$\iint_\Omega \left\{Q_x^*\left(\theta_x+\frac{\partial w}{\partial x}\right)+Q_y^*\left(\theta_y+\frac{\partial w}{\partial y}\right)-\frac{\nu K(\nabla^2 w^*)}{(1-\nu)\lambda^2}w\right\} dx\, dy$$

$$=e_x C_i\left(\theta_x+\frac{\partial w}{\partial x}\right)_i+e_y C_i\left(\theta_y+\frac{\partial w}{\partial y}\right)_i-\frac{2}{(1-\nu)\lambda^2}\oint_\Gamma t_z \nabla^2 w^* ds$$

$$-\frac{2}{(1-\nu)\lambda^2}\iint_\Omega \nabla^2 w^* q\, dx\, dy+\frac{(2-\nu)K}{(1-\nu)\lambda^2}\iint_\Omega w\nabla^2 w^*\, dx\, dy \qquad (5.108)$$

Hence equation (5.104) can be rewritten as follows:

$$C_i\left\{(\theta_x)_i e_x+(\theta_y)_i e_y+w_i e_z\right\}+\oint_\Gamma\left(\theta_x t_x^*+\theta_y t_y^*+wt_z^*\right) ds$$

$$=\oint_\Gamma\left(\theta_x^*\ t_x\ +\ \theta_y^*\ t_y\ +\ w^* t_z+[w^*-\frac{2\nabla^2 w^*}{(1-\nu)\lambda^2}]t_z\right)ds$$

$$+\iint_\Omega qw^*\ dx\ dy-\frac{(2-\nu)}{(1-\nu)\lambda^2}\iint_\Omega (q-Kw)\nabla^2 w^*\ dx\ dy$$

$$+e_x C_i\left(\theta_x+\frac{\partial w}{\partial x}\right)_i+e_y C_i\left(\theta_y+\frac{\partial w}{\partial y}\right)_i \qquad (5.109)$$

Defining θ_1 and θ_2 such that;

$$\theta_1=l_1\theta_x+m\ \theta_y \qquad (5.110a)$$

$$\theta_2=l_2\theta_x+m\ \theta_y \qquad (5.110b)$$

and using an analysis similar to that employed for thin plates, the following boundary integral equations can be deduced from equation (5.109).

$$C_i w_i+\oint_\Gamma\left(T_{11}w+T_{21}\theta_n+T_{31}\theta_t\right)ds$$

$$=\oint_\Gamma\left\{\left(U_{11}-\frac{2\nabla^2 U_{11}}{(1-\nu)\lambda^2}\right)Q_n-U_{21}M_n-U_{31}M_{tn}\right\}ds+\iint_\Omega qU_{11}dxdy+E_1(x_i,y_i) \qquad (5.111a)$$

$$C_i(\theta_x)_i+\oint_\Gamma\left(T_{12}w+T_{22}\theta_n+T_{32}\theta_t\right)ds$$

$$=\oint_\Gamma\left\{\left(U_{12}-\frac{2\nabla^2U_{12}}{(1-\nu)\lambda^2}\right)Q_n-U_{22}M_n-U_{32}M_{tn}\right\}ds+\iint_\Omega qU_{12}dxdy+E_2(x_i,y_i) \qquad (5.111b)$$

$$C_i(\theta_y)_i+\oint_\Gamma\left(T_{13}w+T_{23}\theta_n+T_{33}\theta_t\right)ds$$

$$=\oint_\Gamma\left\{\left(U_{13}-\frac{2\nabla^2U_{13}}{(1-\nu)\lambda^2}\right)Q_n-U_{23}M_n-U_{33}M_{tn}\right\}ds+\iint_\Omega qU_{13}dxdy+E_3(x_i,y_i) \qquad (5.111c)$$

where

$\underset{\sim}{U}$ and $\underset{\sim}{T}$ matrices are given by equations (4.80), (4.81) respectively. Furthermore,

$$E_1(x_i,y_i)=-\frac{(2-\nu)}{(1-\nu)\lambda^2}\iint_\Omega(q-Kw)\nabla^2U_{11}\ dx\ dy$$

$$E_2(x_i,y_i)=-\frac{(2-\nu)}{(1-\nu)\lambda^2}\iint_\Omega(q-Kw)\nabla^2U_{12}\ dx\ dy+C_i\left(\theta_1+\frac{\partial w}{\partial n_1}\right)_i$$

$$E_3(x_i,y_i)=-\frac{(2-\nu)}{(1-\nu)\lambda^2}\iint_\Omega(q-Kw)\nabla^2U_{13}\ dx\ dy+C_i\left(\theta_2+\frac{\partial w}{\partial n_2}\right)_i$$

From thin plate analysis, explicit expressions are given for "U" and "T" from which

$$U_{11}=f=-\frac{1}{2\pi D\kappa^2}K_{ei}(z)$$

where

$$z=\kappa r$$

Hence

$$\nabla^2U_{12}=-\frac{1}{2\pi D}K_{er}(z)$$

i.e.

$$\nabla^2 U_{12} = \frac{\partial}{\partial n_1}(\nabla^2 U_{11})$$

$$= -\frac{\kappa}{2\pi D}\left(\frac{\partial r}{\partial n_1}\right)K'_{er}(z)$$

and

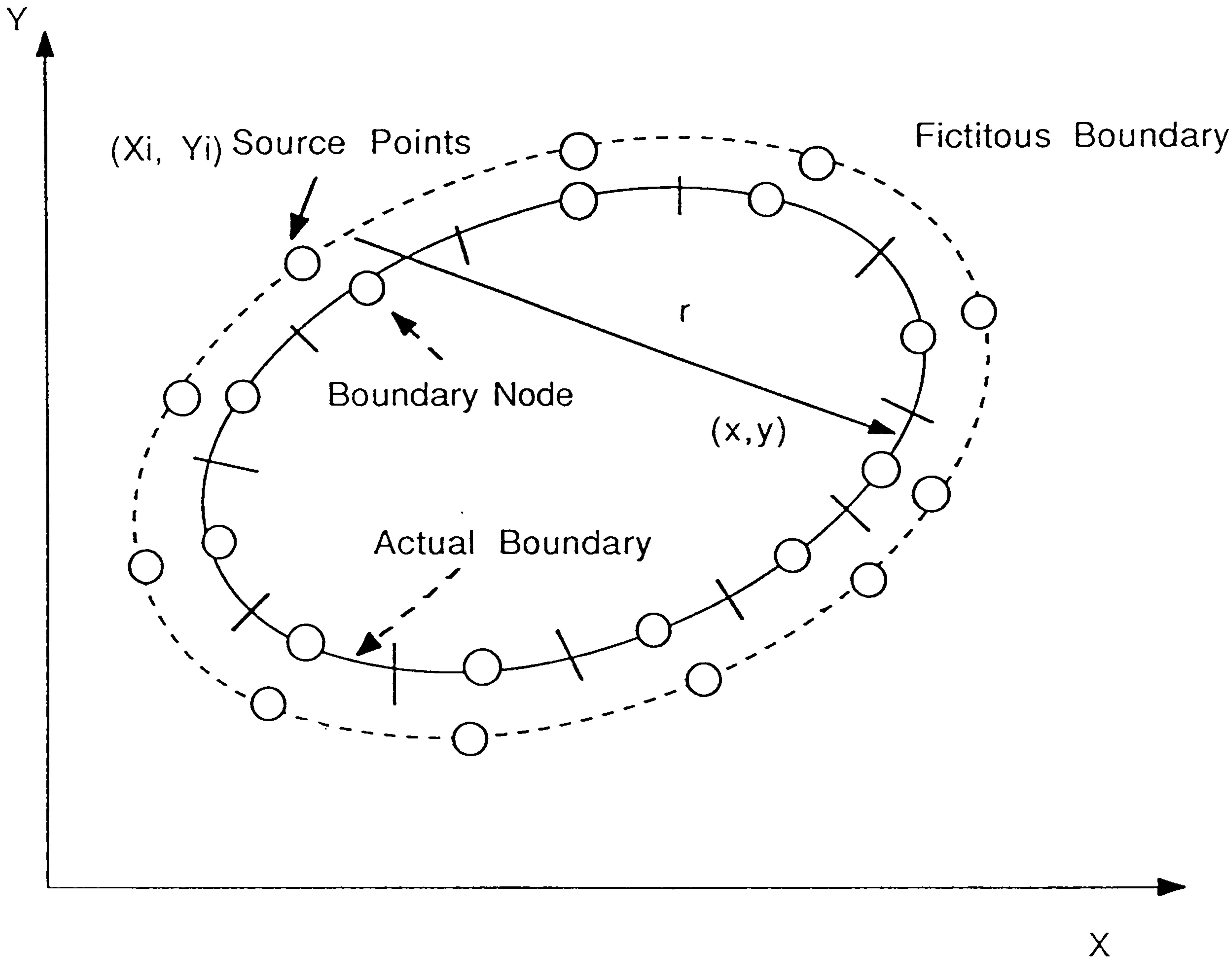
$$\nabla^2 U_{13} = -\frac{\kappa}{2\pi D}\left(\frac{\partial r}{\partial n_2}\right)K'_{er}(z)$$

where

$$\kappa^4 = \frac{K}{D}$$

$$\lambda^2 = \frac{10}{h^2}$$

Correction terms E_1,E_2 and E_3 will be calculated within an iterative procedure.



(Fig. 5.1) Fictitious Boundary

CHAPTER SIX

PROGRAMMING

6.1 INTRODUCTION

In this work, a number of finite element and boundary element programs based upon the theory discussed in chapters 3, 4 and 5 have been designed so as to construct a package, as shown in figure (6.1), for the analysis of plates on elastic foundations. The finite element programs have also been extended to deal with folded plates and facetted shells. A set of simple programs dealing with the analytical solution of thin and thick, circular and rectangular plates on elastic foundations under uniformly distributed and concentrated loading have been also coded and used for validation of developed finite and boundary element programs.

This chapter reviews the basic programs used in the development of the package presented here. The package was called FBEF (Finite Boundary element analysis of plates on Elastic Foundations) and it is demonstrated in figure 6.1.

The development programs are coded in FORTRAN 77 and tested on a mainframe VAX 6000/410 computer. Each program consists of a number of modules, each of which deals with one of the basic steps of the solution procedure.

6.2 FINITE ELEMENT PROGRAMS

The software package contains three finite element programs based upon the theory given in chapter (3). The developed programs have similar structure with many common modules, and they can be summarised as follows:

6.2.1 Kirchhoff element program

This program is a development to an earlier plate bending program written by M. Debbih [Ref 90], and a new subroutine for the derivation of foundation stiffness matrix has been added so as to consider thin plates on elastic foundation. The program contains two Hermitian finite elements; a 3-noded triangular element and a 4-noded quadrilateral element, each with 5-degrees-of-freedom per node so as to consider both in-plane and out-of-plane effects.

The basic structure of the program is as shown in figure 6.2, and program modules are reviewed as follows:

a) Data module

This module consists of DATA subroutine which reads all of the data parameters required for problem definition, such as nodal coordinates, element topology, material and foundation properties and loading and boundary conditions. It calls a subroutine named GAUSS to define the parameters required for Gaussian quadratures.

b) Element stiffness matrix generation module (ESMG)

This module is responsible for the generation of element stiffness matrix and consists of three basic modules:

(i) Subroutine ESMGM which generates the in-plane stiffness matrix with the help of some other subroutines for the calculation of $\underset{\sim}{D}$ and $\underset{\sim}{B}_o$ matrices as defined in section 3.2.4.

(ii) Subroutine ESMGB which generates the out-of-plane stiffness matrix using other subroutines for the evaluation of $\underset{\sim}{D}$ and $\hat{\underset{\sim}{B}}_b$ matrices as defined in section 3.2.4.

(iii) Subroutine ESMGEF which derives the foundation stiffness matrix for each element.

c) Assembler module

In this module, the ASSEMBLER subroutine assembles the finite element equations for the whole structure using element stiffness matrix module (ESMG subroutine). For the case of domain loading a subroutine named ELOAD will be used to calculate the nodal loading vector equivalent to the domain loading.

d) Boundary conditions module

This module consists of one subroutine (REDUCER) which applies the given boundary conditions, and reduces the system of equation to a solvable one. Note that for plates on elastic foundations, it is possible to analyse plates with completely free edge conditions without having singular bending stiffness matrix.

e) Solver module

The reduced system of equations is solved by the SOLVER module which is a subroutine based upon Gauss elimination method.

f) Output module

This module deals with output results and consists of the following basic segments:

(i) DISP subroutine which prints nodal displacements

(ii) REACT subroutine which calculates and prints the reaction forces and moments at restricted nodes.

(iii) STRESS subroutine which calculates the stresses at element nodes. Due to bending effects, the stresses are calculated at the surface $z=+h/2$.

g) Matrix manipulations module

This module consists of subroutines which deal with basic matrices operations as follows:

(i) MATI subroutine which initiates a matrix
(ii) MATM subroutine which multiplies two matrices of compatible dimensions.
(iii) MATV subroutine which multiplies a matrix by a vector.
(iv) MATT subroutine which produces the transpose of a given matrix.
(v) MATS subroutine which sums two matrices.

6.2.2. Mindlin element program

This program is based upon Mindlin's first-order element discussed in section (3.3) using 5 degree-of-freedom per node and considering elastic foundations. The program has a library of five elements which are as follows:

(i) Three-noded triangular element.
(ii) Six- noded triangular element.
(iii) Four-noded quadrilateral element.
(iv) Eight-noded serendipity quadrilateral element.
(v) nine-noded Lagrangian quadrilateral element.

The basic structure of the program is as shown in figure (6.3) which indicates its commonality with the previous program. The program modules are similar to the Kirchhoff program but the fundamental difference is in the ESMG module which consists of a master segment (ESMG) calling the following two basic segments.

a) <u>ESMGIN subroutine</u>

which is the same as ESMGM used in Kirchhoff program and calculates the element in-plane stiffness matrix.

b) <u>ESMGOUT subroutine</u>

Although this subroutine is designed for the calculation of element out-of-plane stiffness matrix it is based upon three other different subroutines:

(i) ESMGB calculates $\underline{K}_b$ as defined by equation (3.57).
(ii) ESMGS calculates $\underline{K}_s$ as defined by equation (3.58).
(iii) ESMGEF calculates the foundation stiffness matrix $\underline{K}_f$ as defined by equation (3.63).

This division was intended in order to allow the use of different quadrature schemes for the numerical evaluation of $\underline{K}_b$ and $\underline{K}_s$ and can therefore introduce the reduced integration concept, suggested by [Ref 131,132] so as to improve the accuracy of Mindlin element for thin plates.

6.2.3. <u>High order shear element program</u>

This program is built on the theory of high-order shear element developed in section 3.4 so as to be employed efficiently for thin and thick plates. It contains the Hermitian elements employed with Mindlin program and each node has seven degrees-of-freedom. The basic modules of the program are similar to the previous finite element programs with many common segments between them. The ESMG module is the one with fundamental difference and it consists of the following segments:

a) ESMGIN subroutine

This subroutine calculates the in-plane element stiffness matrix $\underline{K}_o$ defined in section (3.4.5), as follows:

$$\underline{K}_o = \iint \underline{B}_o^t \underline{D}_o \underline{B}_o \, dx \, dy$$

A subroutine called BMATRIXIN is used to calculate $\underline{B}_b$ matrix using Cartesian derivatives of Lagrangian shape functions, whilst DMATRIXIN subroutine is employed for the evaluation of $\underline{D}_o$ matrix.

b) ESMGB subroutine

This subroutine calculates the element $\underline{K}_b$ matrix defined in section (3.4.5) as follows:

$$\underline{K}_b = \iint \underline{B}_b^t \underline{D}_b \underline{B}_b \, dx \, dy$$

The $\underline{B}_b$ matrix which is based upon second order derivatives of Hermitian shape functions are calculated by means of BMATRIXB and $\underline{D}_b$ is defined in section (3.4.4) and calculated using DMATRIXB.

c) ESMGT subroutine

This subroutine calculates the element $\underline{K}_t$ matrix defined in section (3.4.5) as

$$\underline{K}_t = \iint \underline{B}_o^t \underline{D}_t \underline{B}_o \, dx \, dy$$

Subroutines DMATRIXT and BMATRIXIN are employed to calculate $\underline{D}_t$ and $\underline{B}_o$ respectively.

d) ESMGS subroutine

This subroutine is used to calculate the element stiffness matrix $\underline{K}_s$ defined by

$$\underline{K}_s=\iint \underline{B}_s^t\underline{D}_s\underline{B}_s \, dx \, dy$$

A subroutine called BMATRIXS based upon Lagrangian shape functions is used to calculate $\underline{B}_s$ as defined in equation (3.88), $\underline{D}_s$ is calculated using subroutine DMATRIXS.

e) ESMGBT subroutine

This subroutine is used for the calculation of $\underline{K}_{bt}$ which is defined as

$$\underline{K}_{bt}=\iint \underline{B}_b^t\underline{D}_{bt}\underline{B}_o \, dx \, dy$$

using subroutines BMATRIXIN and BMATRIXB as before, and a subroutine DMATRIXBT is used to calculate $\underline{D}_{bt}$ as defined in section (3.4.4).

f) ESMGEF subroutine

This subroutine calculates the foundation stiffness matrix as defined in section (3.4.6) using Hermitian shape functions.

Notice that the master segment ESMG, the element total stiffness matrix is assembled from the previous matrices according to section (3.4.7)

6.3 BOUNDARY ELEMENT PROGRAMS

6.3.1. programs for the analysis of thin plates on elastic foundations

a) programs using Kelvin functions

These are two programs base upon the fundamental solution expressed in terms of Kelvin functions as described in section (4.2.3). The first program which is referred to as (BEM-THIN-EF-O) uses the domain loading terms derived in section (4.2.5) whilst the second program (BEM-THIN-EF-M) is based upon modified domain loading terms discussed in section (4.3.4). The two programs have identical structure as shown in figure (6.5) and they were based upon modifications of programs written by M. Debbih [Ref 90] but with a different fundamental solution to consider plates on elastic foundations.

The main features of the programs are as follows:

(i) Kelvin functions are calculated for small and large amplitudes using asymptotic expressions given by [Ref 138]

(ii) The stresses, bending moments and shear forces, are calculated at internal nodes by using a finite difference scheme.

(iii) The logarithmic terms within the fundamental solution parameters are integrated by means of analytical expressions whenever the source point occurs on the integration element and this approach has taken care of singular terms.

The basic subroutines of each program are as follows:

(i) DATA reads in the required data, all the BEM programs introduced in this work use constant boundary elements.

(ii) GH calculates the constant element matrices $\underset{\sim}{g}$ and $\underset{\sim}{h}$ at a given source point.

(iii) GHMAT assembles $\underset{\sim}{G}$ and $\underset{\sim}{H}$ matrices for all boundary nodes.

(iv) BC rearranges the matrices according to the given boundary conditions.

(v) SOLVER solves the equations prepared with BC using a Gauss elimination.

(vi) OUTPUT prints the node displacements and tractions for the whole boundary.

(vii) INTERNAL evaluates the required field parameters such as displacements, tractions, stresses and strains at given internal nodes.

(viii) BSTRESS evaluates stress at the boundary points.

(ix) STRESS evaluates the stress at a given internal point.

(x) CLOAD evaluates the reduced load term integrals for the case of a concentrated load.

b) programs using "Modified" Kelvin functions

A disadvantage of the previous program was discovered when the foundation stiffness was taken to be very small or zero. Hence, two other programs were developed using a superposition of plate-bending fundamental solution and the fundamental solution based upon "modified" Kelvin functions as described in section (4.3). The first of them (BEM-THIN-PB-O) employs the modified domain loading terms given in section (4.2.5) and the second (BEM-THIN-PB-M) employs the modified domain loading terms as given in section (4.3.4). The basic structure of these two other programs is shown in figure (6.6)which is similar to the structure of the previous programs shown in figure (6.5) except in the way by which the fundamental solution is defined.

The fundamental solution of plate-bending without foundation has allowed us to calculate the divergent integrals encountered in the boundary integral equations by using rigid translation and rigid rotation conditions whilst the fundamental solution based upon modified Kelvin functions does not contain singular terms.

6.3.2. Thin-thick boundary element programs

This program is based upon the fundamental solution of thin plates using Kelvin functions. Whenever required the effect of thickness is calculated by means of domain integrals as explained in section (5.8) and 3-noded integration cells are used for this purpose. A new idea for overcoming singularities have been introduced in this program by taking the source points corresponding to boundary nodes at a fictitious boundary outside the domain as described in section (5.7). Other features of the program are identical to (BEM-THIN-EF-O) program described earlier.

6.3.3. Thick boundary element program

This program represented a big challenge because it is based on a fundamental solution parameters which are expressed in terms of complex Bessel functions, as explained in chapter (5).

A sophisticated set of subroutines was designed to calculate Bessel functions with real and complex arguments as accurate as possible which are summarised as follows:

(i) For real arguments

Asymptotic expressions are used for small and large arguments as given by [Ref 137] .

(ii) For complex arguments

Two sets of infinite series were employed the first set was used in the evaluation of $K_o(z)$ and $K_1(z)$ where $|z|>25$ and the second set is used when $|z|\leq 25$. The series are written in FORTRAN using double complex variables and the summations continue until the last term is $<10^{-20}$.

Due to the importance of Bessel functions in this boundary element program, a special test program was designed to test them and their values obtained by those subroutines were compared with tables given in [Ref 137]. To test $K_o(z)$ and $K_1(z)$ with z having complex values, the following relationships were used by putting

$z=x(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}})$, in which x is real

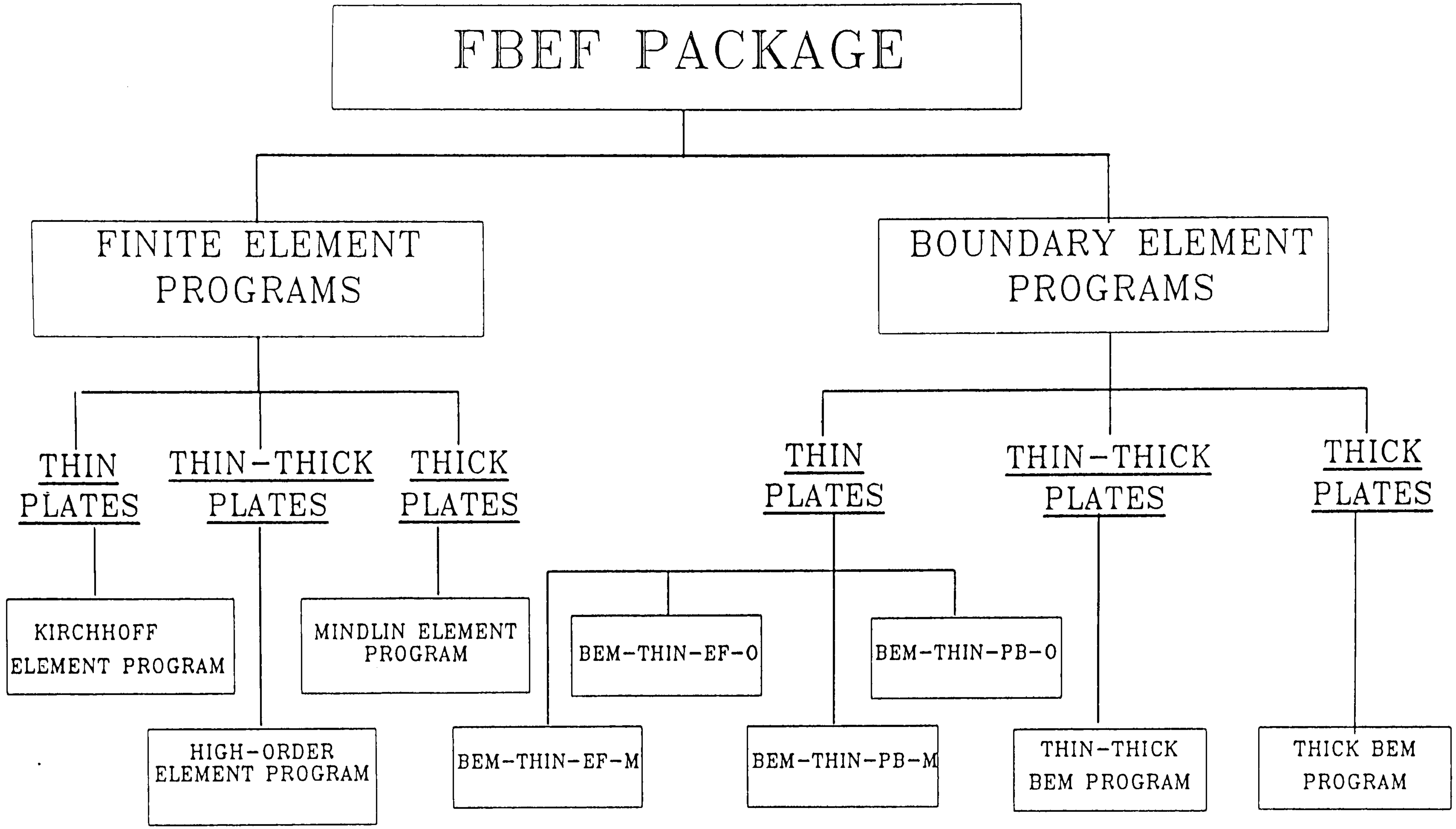
then

$$K_o(z)=K_{er_o}(x)+iK_{ei_o}(x);$$

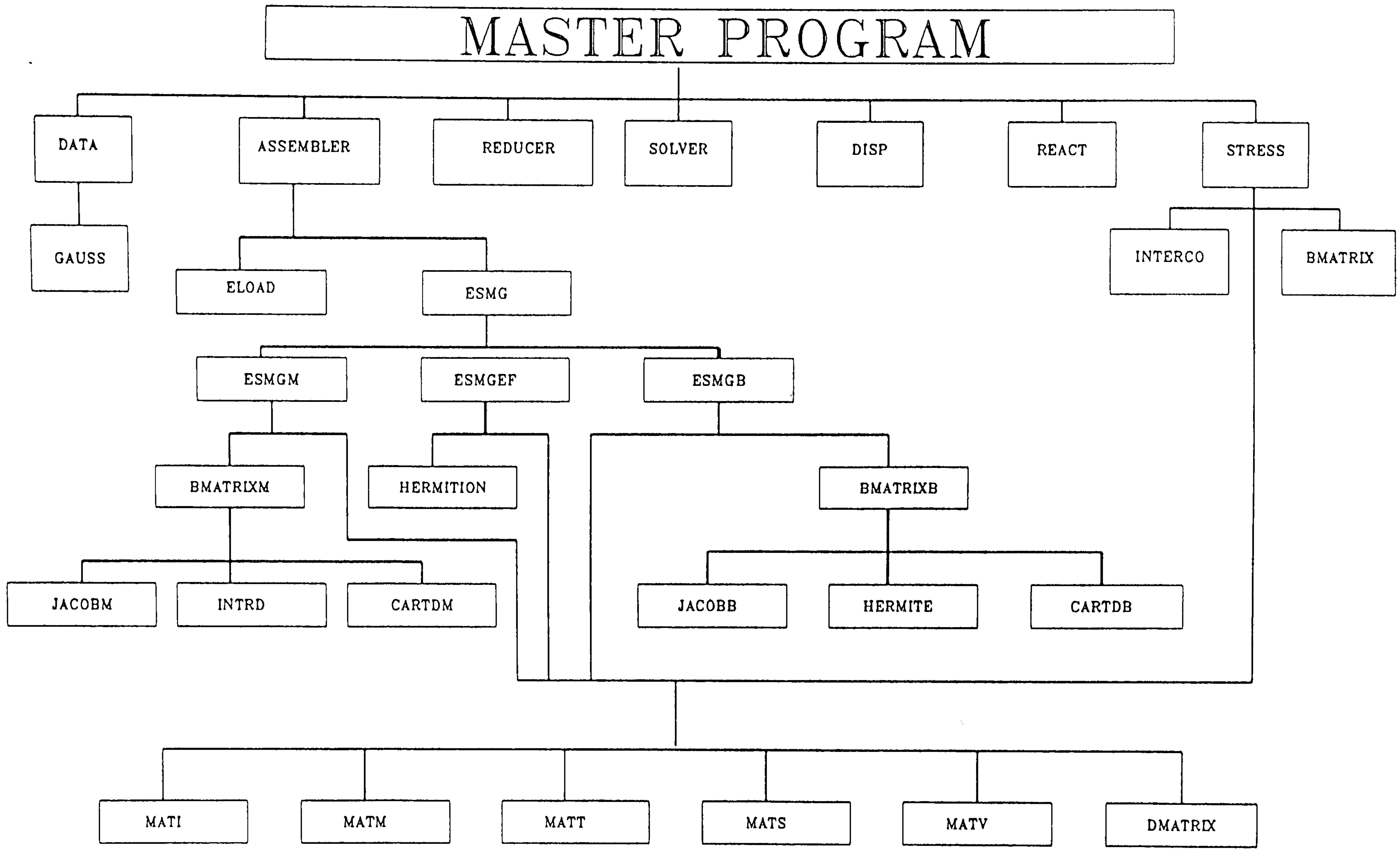
$$K_1(z)=-K_{ei_1}(x)+iK_{er_1}(x);$$

and table for the functions K_{er} and K_{ei} with real arguments are then taken from [Ref 137] for validation.

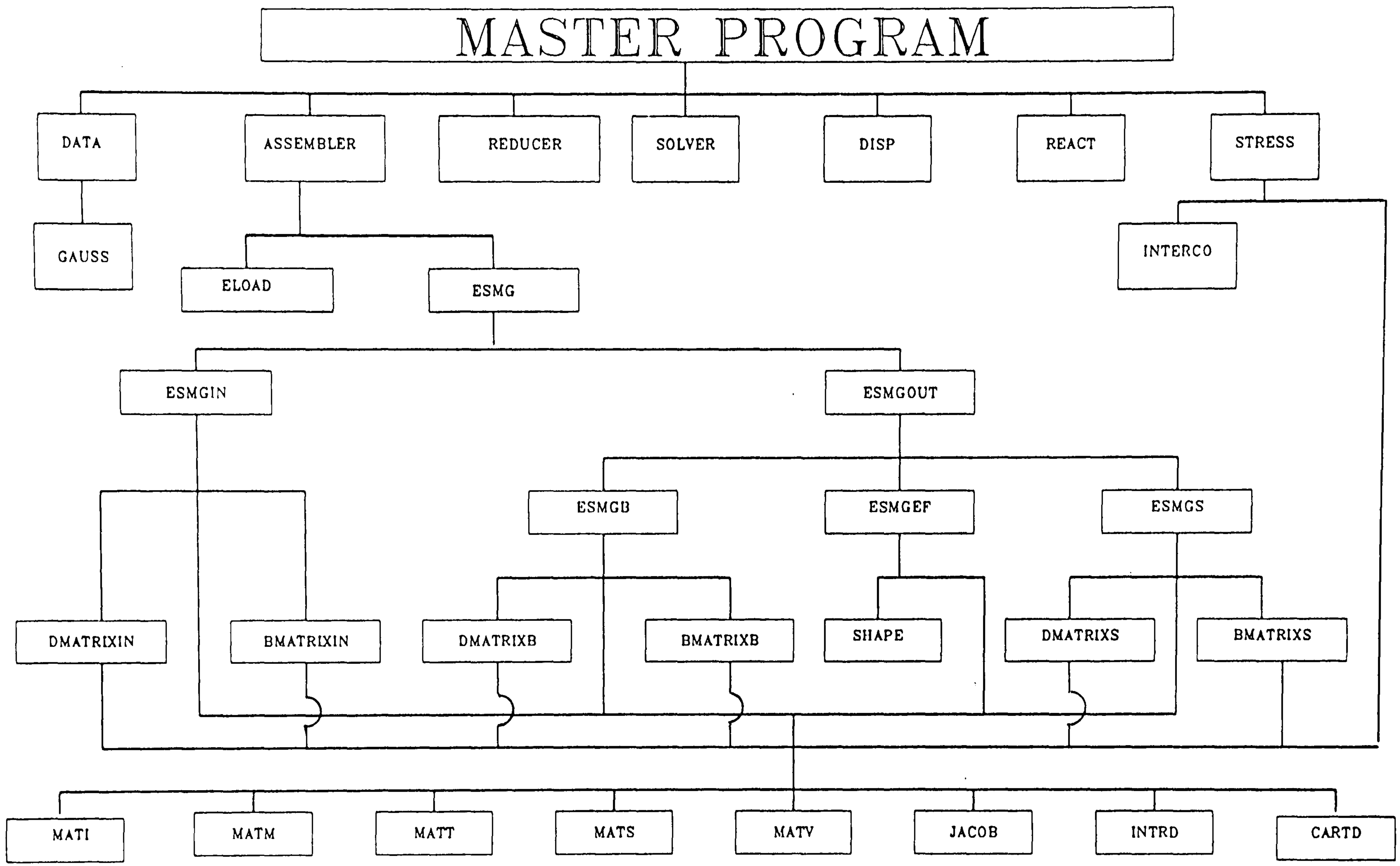
The new idea of using source points on a fictitious boundary outside the domain was also employed in this program to overcome the singular and divergent integrals encountered in the boundary integral equation. The basic structure of the program as shown in figure (6.7) is similar to other boundary element programs.



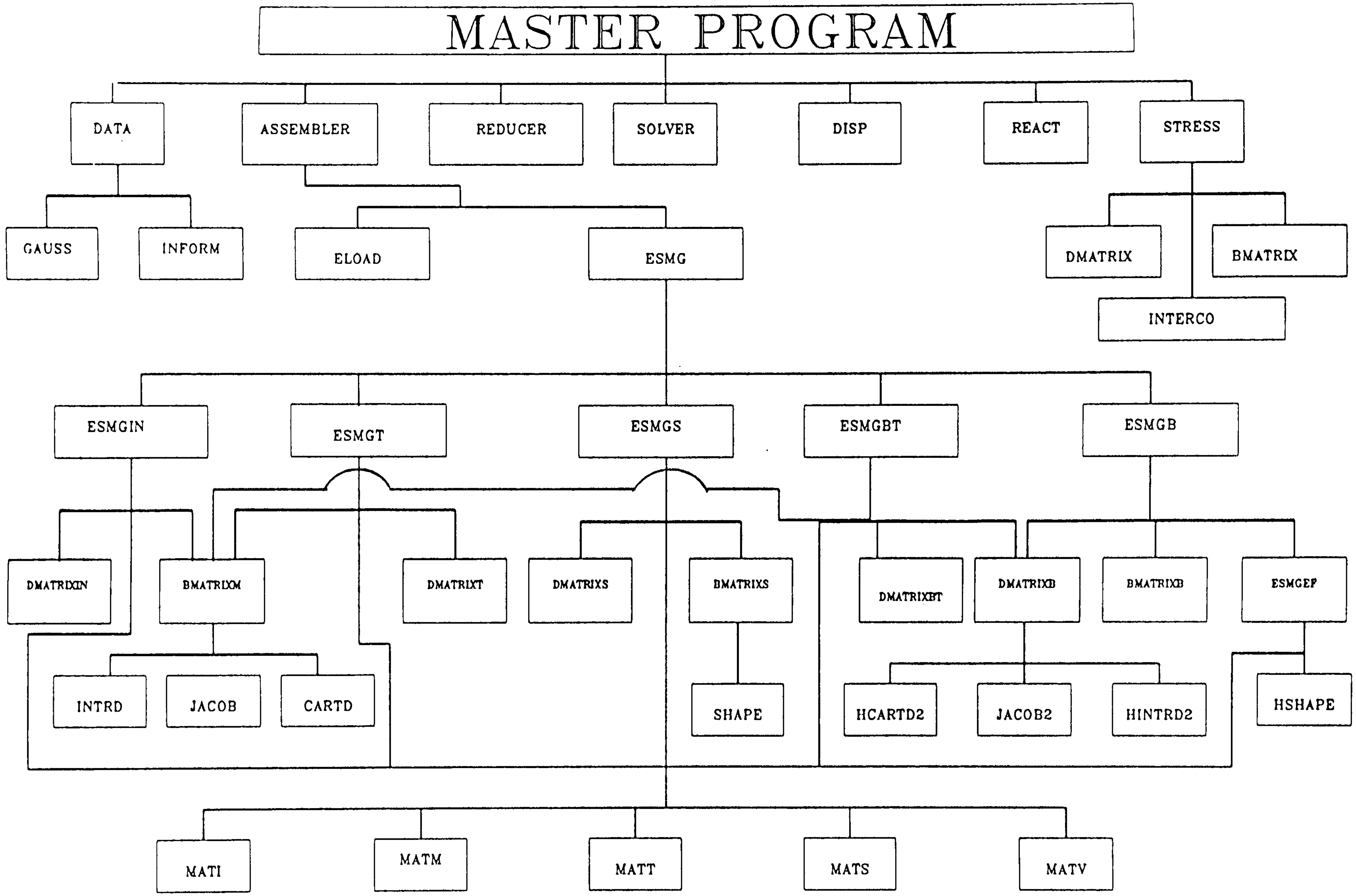
[FIG 6.1] FINITE-ELEMENT, BOUNDARY-ELEMENT PACKAGE FOR THE ANALYSIS OF PLATES ON ELASTIC FOUNDATION



[FIG 6.2] BLOCK DIAGRAM FOR FEM PLATE-KIRCHOFF PROGRAM



[FIG 6.3] BLOCK DIAGRAM FOR FEM PLATE-MINDLIN PROGRAM



[FIG 6.4] BLOCK DIAGRAM FOR FEM HIGH-ORDER SHEAR ELEMENT PROGRAM

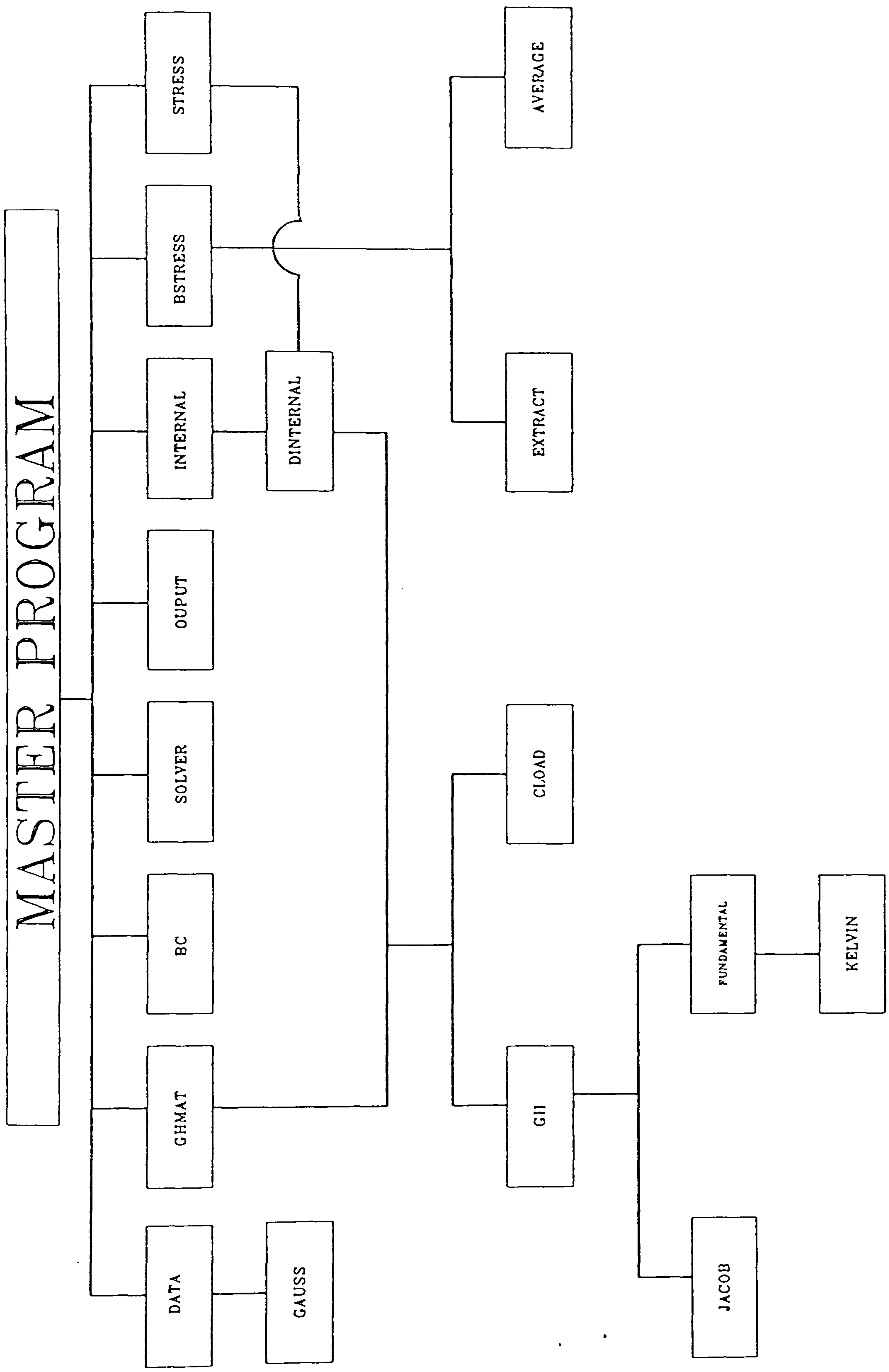
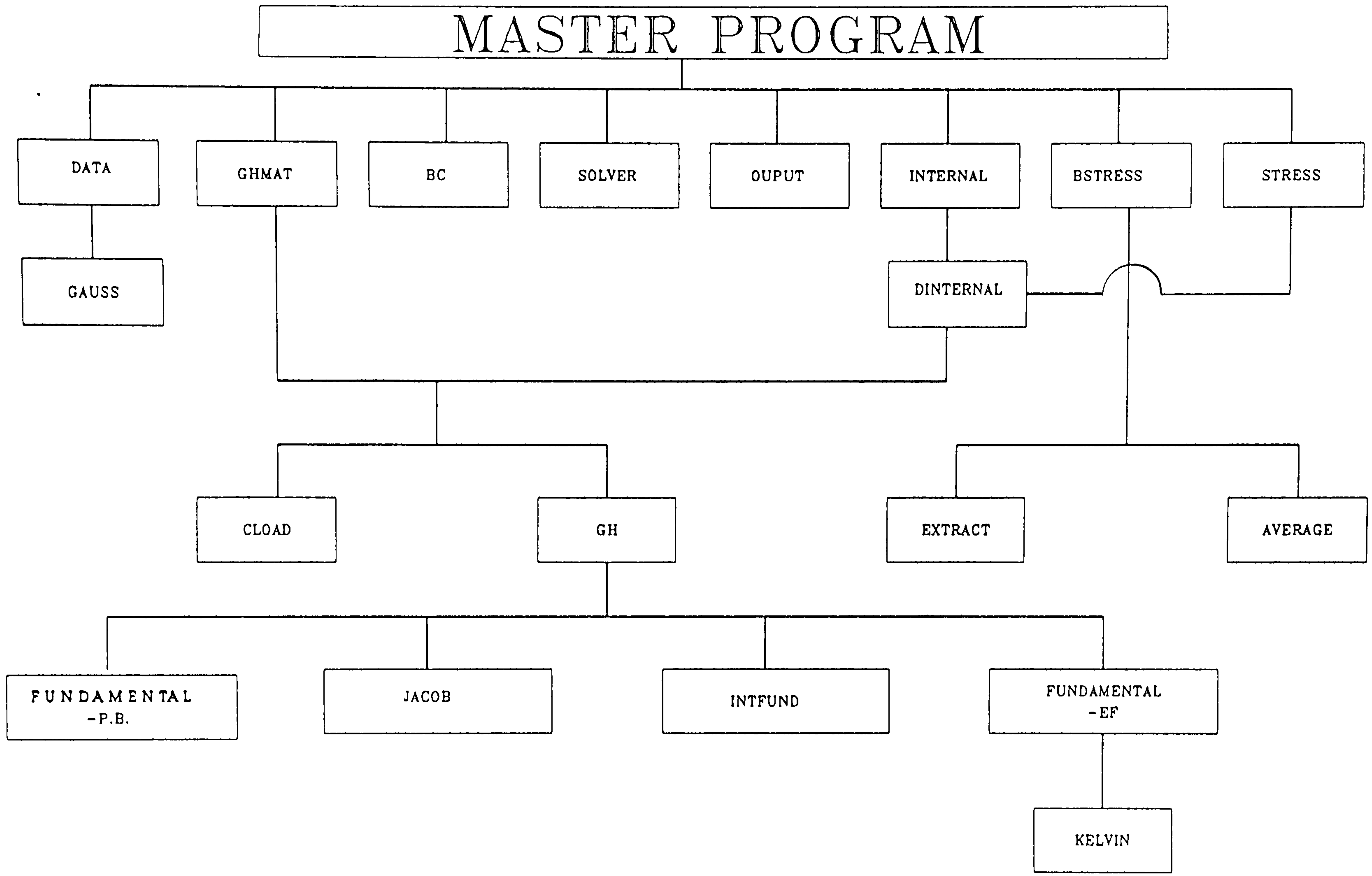
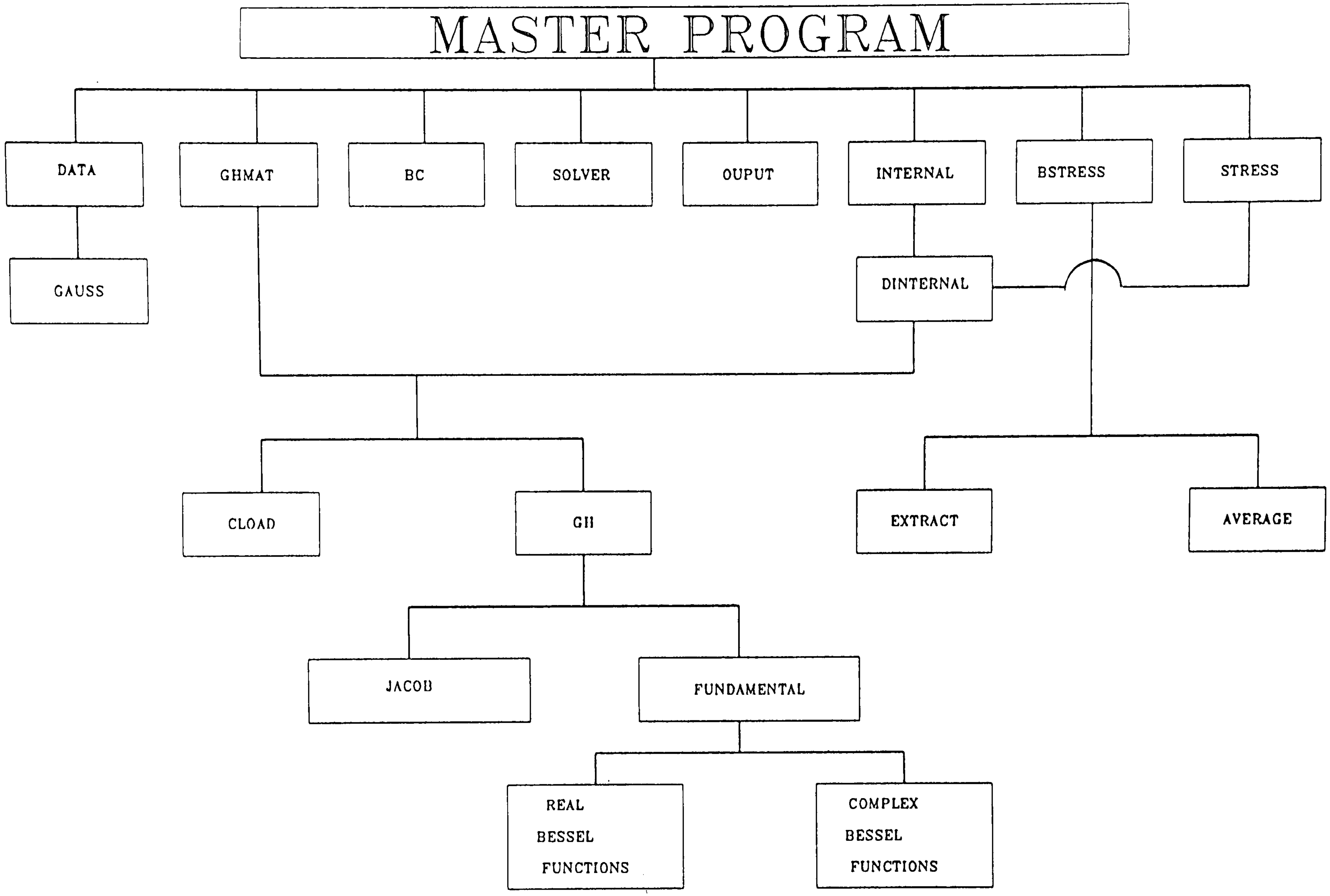


FIG 6.5 BLOCK DIAGRAM FOR BEM–THIN–EF–O/M PROGRAMS



(FIG 6.6) BLOCK DIAGRAM FOR BEM-THIN-PB-EF-O/M PROGRAMS



[FIG 6.7] BLOCK DIAGRAM FOR BEM THICK PROGRAM

CHAPTER SEVEN

RESULTS AND DISCUSSION

7.1 INTRODUCTION

The programming package presented in chapter 6 is based on many original derivations and it was essential to validate the package and verify those derivations by using a range of case studies the solution of which covers all the derivations involved in the analysis and such case studies should have reliable analytical solutions. Circular plate cases were employed because it is possible for such cases with radially symmetric loading to treat their partial differential equations as ordinary differential equations and derive analytical solutions for them. However, they are too easy for the boundary element analysis since they have a uniform geometry without any corners. Hence, a second set of case studies which consists of rectangular plate case with known analytical solutions have been employed.

All of the circular and rectangular plate cases were tested with a uniformly distributed loading and a concentrated loading together with clamped, simply-supported and free edge conditions and the results were plotted against analytical solutions. To reduce the volume of this thesis, samples of results are only presented in this chapter and some other results are given in Appendix (I).

7.2 CASE STUDIES OF THIN PLATES ON ELASTIC FOUNDATION

7.2.1 Clamped circular disc under a uniformly distributed loading

The circular disc employed for this case has the following properties :

Outer radius (a)=10 in
Young's Modulus (E) = 10.92×10^6 psi
Poisson's Ratio (ν) = 0.3
Foundation stiffness (k) = 20,000 lb/in^3
Domain loading intensity (q) = 1000 psi
Plate thickness (h) = 1.0 in

The above values were selected to make to match a similar case published by

Costa and Brebbia [Ref 115]. A boundary element mesh with 40 constant elements, as shown in Figure (7.1), was employed for all circular plate case studies presented in this chapter. Two finite-element meshes were used, the first is a coarse mesh with 25 three-noded triangular elements as shown in Figure (7.2), and the second is a fine mesh with 100 three-noded triangular elements as illustrated in Figure (7.3).

The first test was aimed at validating the 4 different thin-plate boundary element programs developed in this work, as described in chapter 6. The radial distribution of lateral displacement "w" and bending moment (Mr) are plotted against the analytical solution employed by [Ref 115] , as shown in Figures (7.4) and (7.5) which prove that all of those four programs produce similar results which are in good agreement with the corresponding analytical solution.

The finite element lateral displacement results obtained by Kirchhoff-element program are plotted against some of the boundary element results and the analytical solution, as shown in Figure (7.6), which indicates that Kirchhoff 3-noded triangular finite element dose not produce results as accurate as those produced by means of the BEM.

The same case study was tested with different values of foundation stiffness modulus (k) and the values of non-dimensional centre deflection are tabulated versus dimensionless foundation modulus against published results of NG [Ref 121] and Costa and Brebbia [Ref 115] as shown in table (7.1), where

$$\text{Dimensionless Foundation Modulus} = \frac{KD}{a^4}$$

$$\text{Non-dimensional centre deflection} = \frac{W_{centre} D}{qa^4} \times 100$$

It is clear from that table that our results are nearer to the analytical solution given by [Ref 121] than those given by Costa and Brebbia [Ref 115].

7.2.2 Simply-supported disc under a uniformly distributed loading

For the cases of clamped plates, only the $\underset{\sim}{G}$ matrix and domain loading terms are required for the analysis. Hence, to test other derivations another case similar to the previous one was employed but with a simply-supported edge. The same boundary-element and finite element meshes are used. The radial distribution of the lateral displacement "w" and bending moment "Mr" as obtained by means of the 4 thin-plate boundary element programs developed in this work are plotted against the analytical solution used by Costa and Brebbia [Ref 115], as shown in Figures (7.7) and (7.8). It is clear from those figures that our boundary element programs give identical results which are in good agreement with the results obtained by means of the analytical solution. The lateral displacement obtained by using Kirchhoff-element program for the coarse and fine meshes, shown in Figures (7.2) and (7.3), are plotted against BEM results and analytical solution in Figure (7.9) which indicates a phenomenon similar to that observed in Figure (7.6).

7.2.3 Cases of clamped rectangular plates under a uniformly distributed loading

One of the advantages of the three degrees-of-freedom derivation presented in this work is the elimination of Kirchhoff corner forces encountered with two degrees-of-freedom derivations. Hence, it is essential to test case studies with corners such as rectangular plates to verify our claims.

The example presented in this section was carried out for a series of clamped rectangular plates with different aspect ratios. The parameters used for the result presentation of the plate shown in figure (7.10) are defined as follows:

Plate aspect ratio $(\lambda) = \frac{b}{a}$

Deflection coefficient $(\bar{w}) = \frac{W_{max} \cdot D}{qa^4} \times 100$

Non-dimensional foundation stiffness $(\bar{K}) = \frac{KD}{a^4}$

Centre moment coefficent $(C) = \frac{M_c}{qa^2} \times 100$

Edge moment coefficient $(m) = \frac{M_e}{qa^2} \times 100$

where

$M_c =$ Moment at the plate centre.

M_e= Maximum edge moment.

The deflection coefficient was plotted against aspect ratio, as shown in Figure (7.11) which proves that our results are in good agreement with the analytical solution used by [Ref 115]. Centre moment and edge moment coefficients are also plotted and compared with the analytical solution as demonstrated in Figure (7.12) which emphasis the accuracy of our derivations.

7.2.4 Cases of simply-supported rectangular plates

As mentioned in section (7.2.2), simply-supported cases are useful for testing other derivations including corner forces effect which dose not exist for cases with clamped edges. Analytical solutions for thin simply-supported rectangular plates on elastic foundation, under domain and concentrated loading are developed using a double Fourier series similar to that given by Frederick [Ref 105], and summarised in Appendix(G).

A rectangular plate, as shown in Figure (7.13) with material and foundation properties similar to that used in the previous case, and with A=10 in, B=5 in, h=1.0 in.was used in this study. The boundary element mesh and two finite element meshes (coarse and fine) used in this example are as shown in Figures (7.14), (7.15) and (7.16), respectively.

The first case tested was with uniform domain loading of intensity (q)=1000psi, and the x-axis distribution of displacement (w) and slope ($\partial w/\partial x$) along the centre line of plate are plotted against the analytical solution as shown in Figure (7.17) and (7.18), respectively which indicate an excellent agreement with the analytical solution for the results obtained by boundary element and finite element programs.

The second case was with a concentrated force F=-10lb acting at the plate centre, and the deflection (w) and slope ($\partial w/\partial x$) were plotted in a way similar to the previous case and shown in Figures (7.19) and (7.20). The results obtained for the FEM coarse mesh shows some deviation from the analytical solution results in the slope figure. It is also worth mentioning that it is not possible in boundary programs to obtain results at the exact point at which the load is applied due to a singularity developed in the fundamental solution parameters.

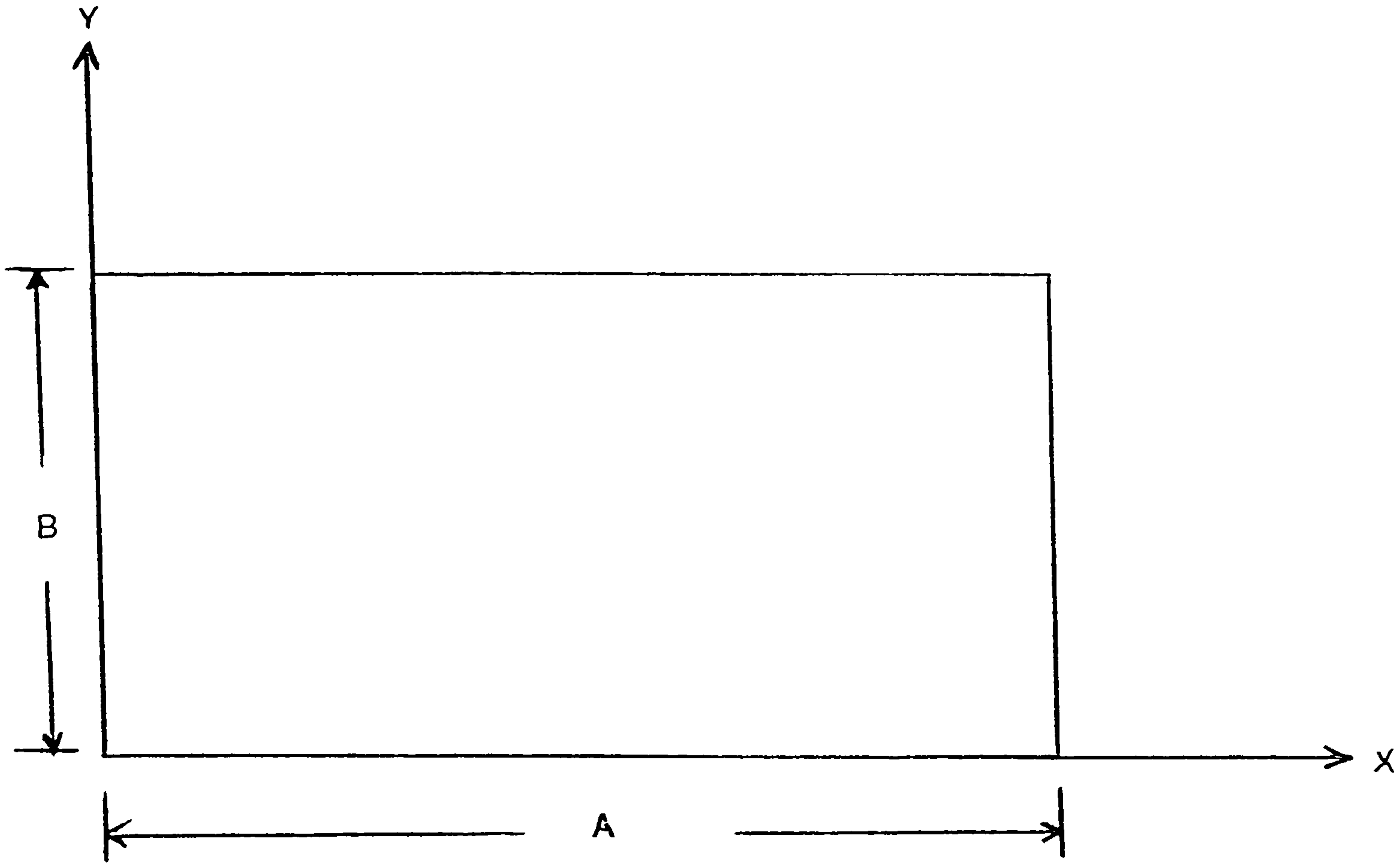


Figure [7.13] Rectangular plate for the simply-supported case.

$$w(x,y)=\sum_m\sum_n w_{mn}\,\sin(\alpha_{mx})\,\sin(\beta_{ny})$$

$$\alpha_m=\frac{m\pi}{A} \qquad \beta_n=\frac{n\pi}{B}$$

7.3 CASE STUDIES OF THICK PLATES ON ELASTIC FOUNDATION

7.3.1 Cases of a circular disc under a uniformly distributed loading

The circular disc described in section (7.2.1) was employed in this study in which different boundary conditions together with two different thicknesses (h1=1in, h2=2in) were tested. The same boundary element mesh, as shown in Figure (7.1), and the finite element coarse mesh shown in Figure (7.2), were employed for this study. The programs used for the analysis are as follows :

(a) BEM Thin which is the thin version of the fifth boundary element program for the analysis of thin and thick plates on elastic foundation and it is based upon using a fictitious boundary for the elimination of singularities, as explained in chapter 6.

(b) BEM FACET-HER, which is the newly developed program for thick plates on elastic foundation

(c) FEM FACET-HER, which is the finite element program based upon high-order shear element.

The analytical solution for thin and thick circular plates on elastic foundation under uniformly-distributed loading has been developed by the current author and described in Appendix (G), from which it was found that :

(a) <u>For thin plates :</u>

$$w(r)=c_1\, b_{ei}(\kappa r)+c_2\, b_{er}(\kappa r)+\frac{q}{K}$$

where

$$\kappa=\left(\frac{K}{D}\right)^{0.25}$$

c_1, c_2 are integration constants obtained from the boundary conditions.

(b) <u>For thick plates:</u>

$$w(r)=c_1\, I_o(\lambda_1 r)+c_2\, I_o(\lambda_2 r)+\frac{q}{K}$$

where

$$\lambda_1^2=b+\sqrt{b^2\text{-}c}$$

$$\lambda_2^2=b-\sqrt{b^2\text{-}c}$$

$$b=\frac{(2-\nu)}{2(1-\nu)}\cdot\frac{K}{\lambda^2 D}$$

$$c=\frac{K}{D}$$

A simple program was developed to calculate such analytical solutions for clamped, simply-supported, and free edge circular plates.

The results obtained for different cases can be summarised as follows :

(a) Clamped Circular Disc:

Radial displacement and slope distributions are shown for (h=1in) in figures (7.21), (7.22) and for (h=2in) in Figures (7.23), and (7.24), respectively.

It is clear from these figures that :

(I) All the programs give very similar results in the thin range

(II) In the thick range, there is a significant difference between the results obtained by means of thin and thick programs due to transverse shear effects.

(III) The newly developed thick boundary element program results are very close to the thick analytical solution.

(IV) The displacement obtained by high-order 3-noded element is in good agreement with the thick analytical solution. However, some deviation of the FEM results observed in the slope curves due to the coarse mesh employed.

(b) Simply-Supported Circular Disc

The displacement and slope results for this case are shown in figures (7.25) to (7.28) which display phenomena similar to those experienced with the previous case.

(c) Free-Free Circular Disc

It was not possible to obtain acceptable results for discs with free edge conditions using the first four thin boundary element programs due to the effect of terms with divergent integrals. When the analytical solution was developed, it was found that for such a case $C1 = C2 = 0$, and

$$w(r) = \frac{q}{K} \qquad \text{For all thicknesses.}$$

i.e the plate stiffness has no effect on the lateral displacement.

The first successful boundary element results for such a case were obtained when the idea of a fictitious boundary was introduced in the thin and thick boundary element programs and it was found that at all internal nodes,

$$w(r) = 0.05 \equiv \frac{q}{K} \quad \text{For } h=1,\ h=2$$

The finite element results gave an average value of the same order but with small variation as shown in Figure (7.29). The corresponding slope angles are with negligible values as shown in Figure (7.30).

7.3.2 Cases of a circular disc under a central loading (F)

These cases become interesting after an analytical solution was developed, as explained in Appendix (G), where it can be seen that:

(a) For thin Plates:

$$w(r)=c_1\, b_{ei}(kr)+c_2\, b_{er}(kr)-\frac{F}{2\pi D\kappa^2}K_{ei}(kr)$$

(b) For thick plates

$$w(r)=c_1\, I_o(\lambda_1 r)+c_2\, I_o(\lambda_2 r)-\phi_1 K_o(\lambda_1 r)+\phi_2 K_o(\lambda_2 r)+$$

where

κ, c_1, c_2, λ_1, λ_2 are as explained in the previous cases,

$$\phi_1=\frac{F}{4\pi D\sqrt{b^2-c}}\left[1-\frac{(2-\nu)}{(1-\nu)}\left(\frac{\lambda_1}{\lambda}\right)^2\right]$$

$$\phi_2=\frac{F}{4\pi D\sqrt{b^2-c}}\left[1-\frac{(2-\nu)}{(1-\nu)}\left(\frac{\lambda_2}{\lambda}\right)^2\right]$$

$$\lambda^2=\frac{10}{h^2}$$

Notice that the analytical solution for thick plates is singular at the disc centre (r=0), the point of load application. Another simple program was developed for the analytical solution of thin and thick circular plates under a central concentrated loading and with clamped, simply-supported, and free edge conditions.

Radial displacement distribution for thin (h=1) and thick (h=2) discs with clamped, simply-supported, and free edge conditions are demonstrated in Figures (7.31) to (7.36), from which it can be concluded that:

(i) Results obtained by the thin BEM program are very close to thin analytical solution for all cases with different edge conditions.

(ii) Finite element results agree with the thick analytical solution but not as accurate as the BEM results.

(iii) In the zone close to the disc centre (r=0), some artificial deviation between the results can be observed because of the singularity there. The BEM programs can not predict results at the exact location of the concentrated load.

7.3.3 Cases of simply-supported rectangular plate under uniformly distributed loading

A simply-supported rectangular plate similar to that shown in Figure (7.13) with (A = B = 40in) has been considered with 6 different foundation thicknesses (h=2, 4, 8, 12, 16, 20 in) and three different foundation stiffness modulii (K= 200, 2000, 2000 ib/in) and it is similar to a case published by Voyiadjis and Kattan [Ref 107].

Other properties are as follows :

Young's modulus (E) = 30×10^6 psi

Poisson's ratio (ν) = 0.3

Domain loading intensity (q) = 10000 psi

However, no comparison was made here with the tabulated results given in [Ref 107], after finding that it displayed only the results of one-term solutions. The full analytical solution for thin and thick simply-supported rectangular plate was developed in appendix (G), and a simple program was written to calculate the corresponding thin and thick analytical solution.

The maximum deflection of the plate (at its centre) is tabulated for all cases mentioned above, as shown in tables (7.2, (a), (b), (c)) which indicate a reasonable stability of the results in the wide range of thickness and foundation stiffness modulii used.

The radial distribution of deflection has been plotted for the 18 different cases and the plots at h = 2 and h = 12 with k = 200, 2000, 20000 are presented in figures (7.37) to (7.42), whilst the meshes used and the other plots are given in Appendix (H).

It is clear from those figures that

(i) Thin BEM results are in good agreement with analytical solution in the whole range of thickness and foundation stiffness modulli.

(ii) Thick BEM results tend to be slightly higher than analytical results when the thickness increases

(iii) Thick FEM results tends to be lower than analytical results when the thickness increases.

7.3.4 Case of simply-supported rectangular plate under a central concentrated loading

The different cases of the previous section were used with a force F = 100 lb acting at the plate centre. Thin and thick analytical solutions are developed as shown in appendix (G). The maximum displacement of the plate is given in tables (7.3 (a), (b), (c)) which show the stability of the results for such a case.

A sample of the results illustrating the radial distribution of deflection at h = 2 and h = 12, k=200,2000 and 20000 are given in Figures (7.43) to (7.48), and other figures are shown in Appendix (H). It is clear from those figures that general remarks similar to those noticed with the figures of the previous case can be concluded.

7.4 GENERAL DISCUSSION

During the course of this study and from the analysis of previous results, some general interesting points have been observed and can be summarised as follows :

(a) BEM analysis of thin plate on elastic foundation

It is clear that measures taken for dealing with singular and divergent integrals have worked properly and led to accurate answers, for plates with clamped and simply-supported edge conditions. The programs BEM-THIN-PB-O,-M are capable of predicting results for foundation stiffness modulus $K \rightarrow 0$, while other programs (BEM-THIN-EF-O,-M) fail to do so.

The 3 degrees-of-freedom derivations have led to simple programming and more accurate results compared with published 2 degrees-of freedom derivations, and the evaluation of Kirchhoff corner forces is avoided.

(b) Finite-element versus boundary element

It is clear from many of the case studies tested that for the same boundary divisions, the boundary element method proved to be more accurate than the finite element method. The simple 3-noded and 4-noded finite elements were used in the comparison with constant boundary elements. In early tests it was found that such simple finite element are not reliable to be employed with Mindlin first order finite element program which requires the use of parabolic and cubic elements in order to produce an acceptable degree of accuracy.

(c) BEM analysis of thick plates

The program based on the new derivation presented in this work has produced very accurate results. The alternative idea of using thin fundamental solution together with domain integral terms has produced less accurate results due to the accumulation of errors in the numerical integration over the domain. The concept of using a fictitious boundary in thin and thick BEM programs has proved to be very useful in overcoming singular and divergent integral problems encountered with boundary integral equation. It has proved to be reliable in dealing with plates with free edge conditions, which other boundary element programs have failed to deal with.

Dimensionless foundation modulus	Results from B.E.M. (EF)	Results from B.E.M (P.B)	Results from NG [Ref. 121]	Results from Costa [Ref 115]
1	1.531	1.531	----------	---------
2	1.516	1.516	----------	---------
3	1.502	1.502	----------	---------
4	1.488	1.488	----------	---------
5	1.473	1.473	----------	---------
10	1.407	1.407	----------	---------
15	1.346	1.346	----------	---------
20	1.290	1.290	1.301	1.279
100	0.764	0.0764	0.768	0.760
140	0.631	0.631	0.633	0.628
200	0.497	0.497	0.498	0.495

Table [7.1] Non-dimensional centre deflection results.

	Centre deflection $w \times 10^{-4}$				
h(in)	Thin analytical solution	B.E.M Thin	Thick analytical solution	B.E.M Thick	F.E.M (Facet-Her)
2	44587.18	44523.0	45044.51	46933.00	45420.00
4	5869.894	5862.1	6126.026	6618.00	6378.00
8	738.64	737.778	868.598	980.21	932.40
12	219.00	218.77	305.8	347.91	319.10
16	92.407	92.312	157.51	176.79	152.10
20	47.315	47.266	99.405	109.03	85.95

Table [7.2 (a)] Centre deflection of a simply supported rectangular plate on an elastic foundation under uniformly distributed loading (a) case with K=200 lbs/in^3.

	Centre deflection $w \times 10^{-4}$				
h(in)	Thin analytical solution	B.E.M Thin	Thick analytical solution	B.E.M Thick	F.E.M (Facet-Her)
2	29213.92	29183.0	29384.04	30170.0	29320.0
4	5493.992	5486.0	5714.916	6140.4	5925.0
8	732.3469	731.36	859.6897	968.86	922.0
12	218.4474	218.18	304.6387	346.44	317.8
16	92.30823	92.199	157.2021	176.4	151.8
20	47.28936	47.236	99.28018	108.88	85.86

Table [7.2 (b)] Centre deflection of a simply supported rectangular plate on an elastic foundation under uniformly distributed loading (b) case with K=2000 lbs/in^3.

	Centre deflection $w \times 10^{-4}$				
h(in)	Thin analytical solution	B.E.M Thin	Thick analytical solution	B.E.M Thick	F.E.M (Facet-Her)
2	6040.84	6039.7	6018.42	6042.1	5949.0
4	3328.325	3325.1	3393.933	3537.5	3437.0
8	674.239	673.75	779.462	868.1	829.4
12	213.0293	212.73	293.72	322.3	306.1
16	91.3270	91.203	154.1876	172.61	149.1
20	47.03055	46.969	98.050	107.40	84.98

Table [7.2 (c)] Centre deflection of a simply supported rectangular plate on an elastic foundation under uniformly distributed loading (c) case with K=20000 lbs/in^3.

Centre deflection $w \times 10^{-4}$					
h(in)	Thin analytical solution	B.E.M Thin	Thick analytical solution	B.E.M Thick	F.E.M (Facet-Her)
2	8.0221	8.00	8.4221	8.640	8.447
4	1.0487	1.05	1.2531	1.303	1.191
8	0.13184	0.13184	0.13184	0.24164	0.1770
12	3.9089e-2	3.9088e-2	0.10743	0.10804	6.069e-2
16	1.6493e-2	1.6493e-2	6.7752e-2	6.6592e-2	2.876e-2
20	8.4448e-3	8.4448e-3	4.945e-2	4.7777e-2	1.612e-2

Table [7.3 (a)] Centre deflection of a simply supported rectangular plate on an elastic foundation under central concentrated loading (a) case with K=200 lbs/in^3.

Centre deflection $w \times 10^{-4}$					
h(in)	Thin analytical solution	B.E.M Thin	Thick analytical solution	B.E.M Thick	F.E.M (Facet-Her)
2	5.6337	5.6337	5.9871	6.0514	5.884
4	0.99045	0.99045	1.1892	1.23	1.119
8	0.13087	0.13087	0.23293	0.23990	0.1754
12	3.9033e-2	3.9003e-2	0.10725	0.1078111	6.049e-2
16	1.6477e-2	1.6477e-2	6.770e-2	6.653e-2	2.871e-2
20	8.441e-3	8.441e-3	4.943e-2	4.7746e-2	1.611e-2

Table [7.3 (b)] Centre deflection of a simply supported rectangular plate on an elastic foundation under central concentrated loading (b) case with K=2000 lbs/in^3.

Centre deflection $w \times 10^{-4}$					
h(in)	Thin analytical solution	B.E.M Thin	Thick analytical solution	B.E.M Thick	F.E.M (Facet-Her)
2	1.916	1.916	2.22	2.191	2.022
4	0.6537	0.637	0.82695	0.8285	0.7232
8	0.12194	0.22037	0.22037	0.22439	0.1607
12	3.8163e-2	3.8163e-2	0.10552	0.10562	5.862e-2
16	1.6326e-2	1.6326e-2	6.7217e-2	6.5929e-2	2.828e-2
20	8.4007e-3	8.4007e-3	4.923e-2	4.7506e-2	1.597e-2

Table [7.3 (c)] Centre deflection of a simply supported rectangular plate on an elastic foundation under central concentrated loading (c) case with K=20000 lbs/in^3.

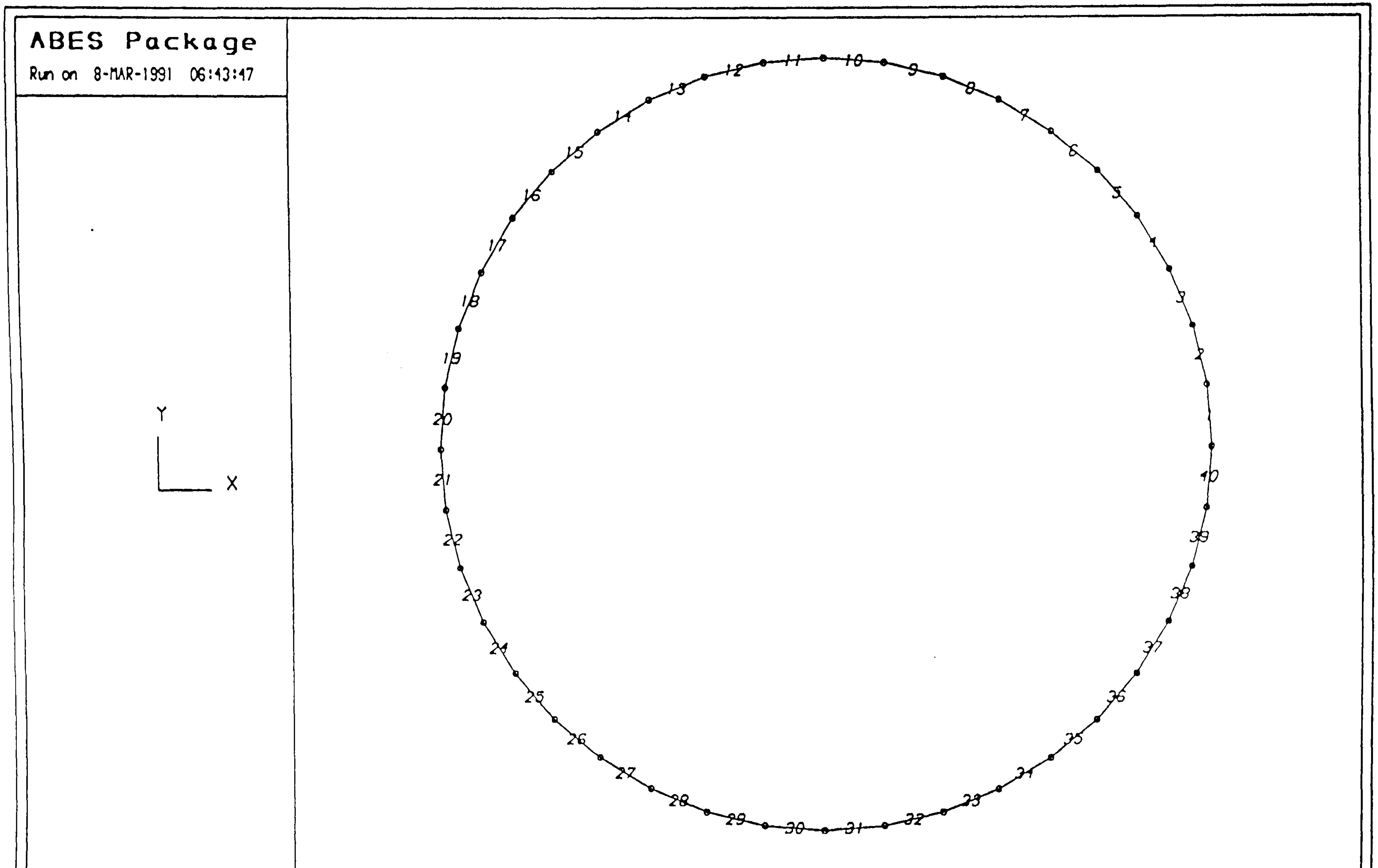


FIG [7.1] BOUNDARY ELEMENT MESH FOR CIRCULAR PLATE CASE STUDIES

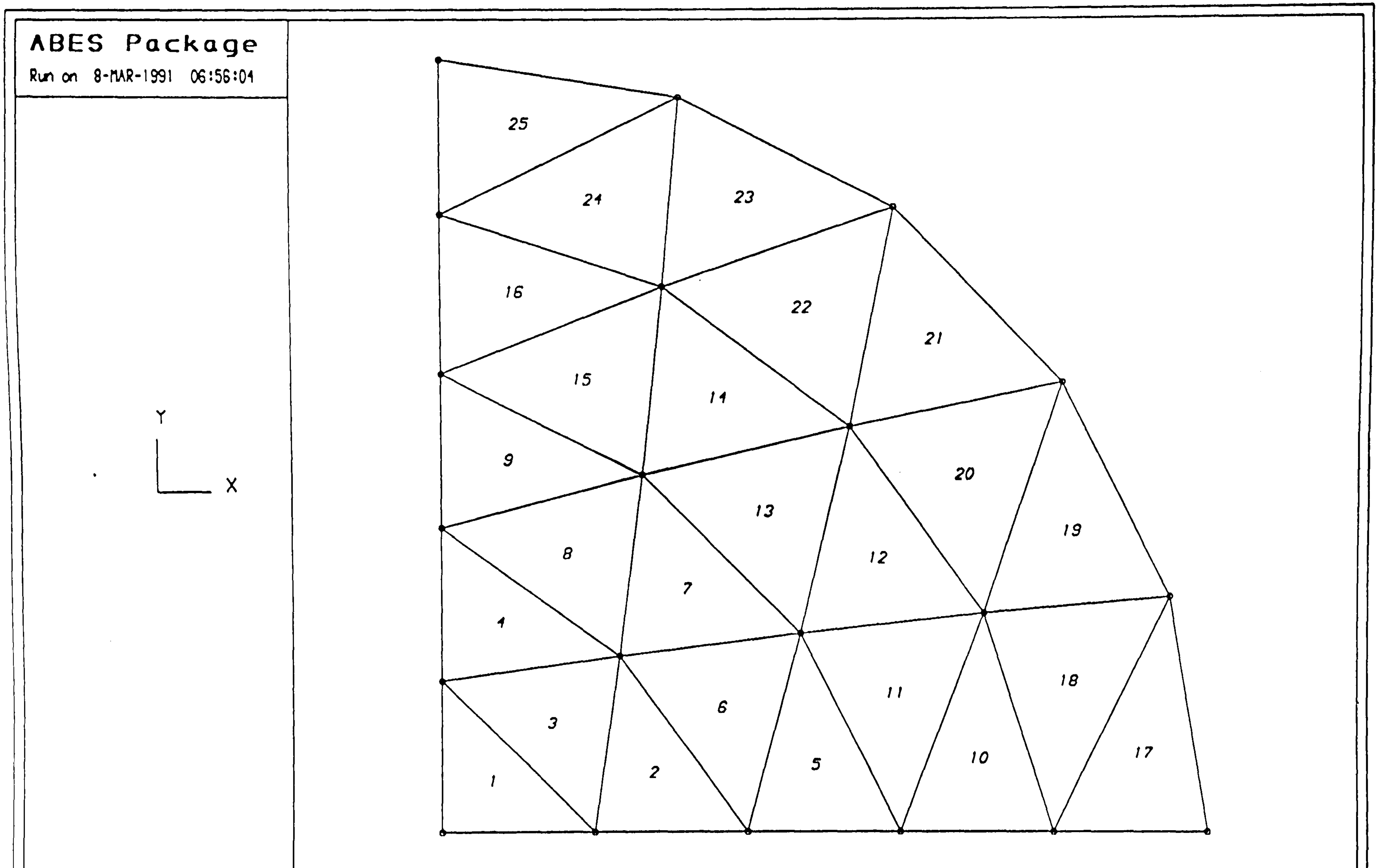


FIG [7.2] COARSE FINITE ELEMENT MESH FOR CIRCULAR DISC CASE STUDIES

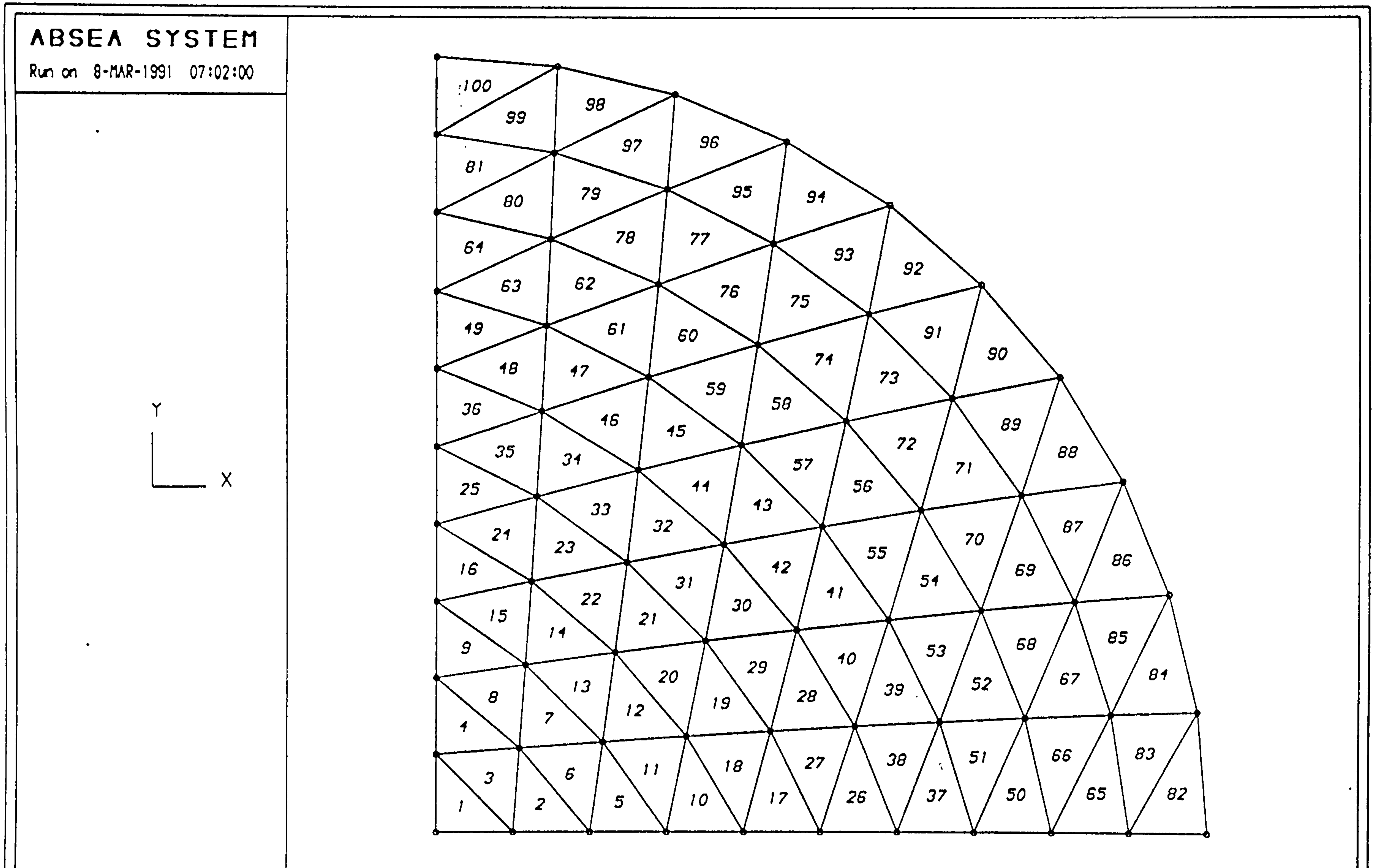


FIG [7.3] FINE FINITE ELEMENT MESH FOR
CIRCULAR DISC CASE STUDIES

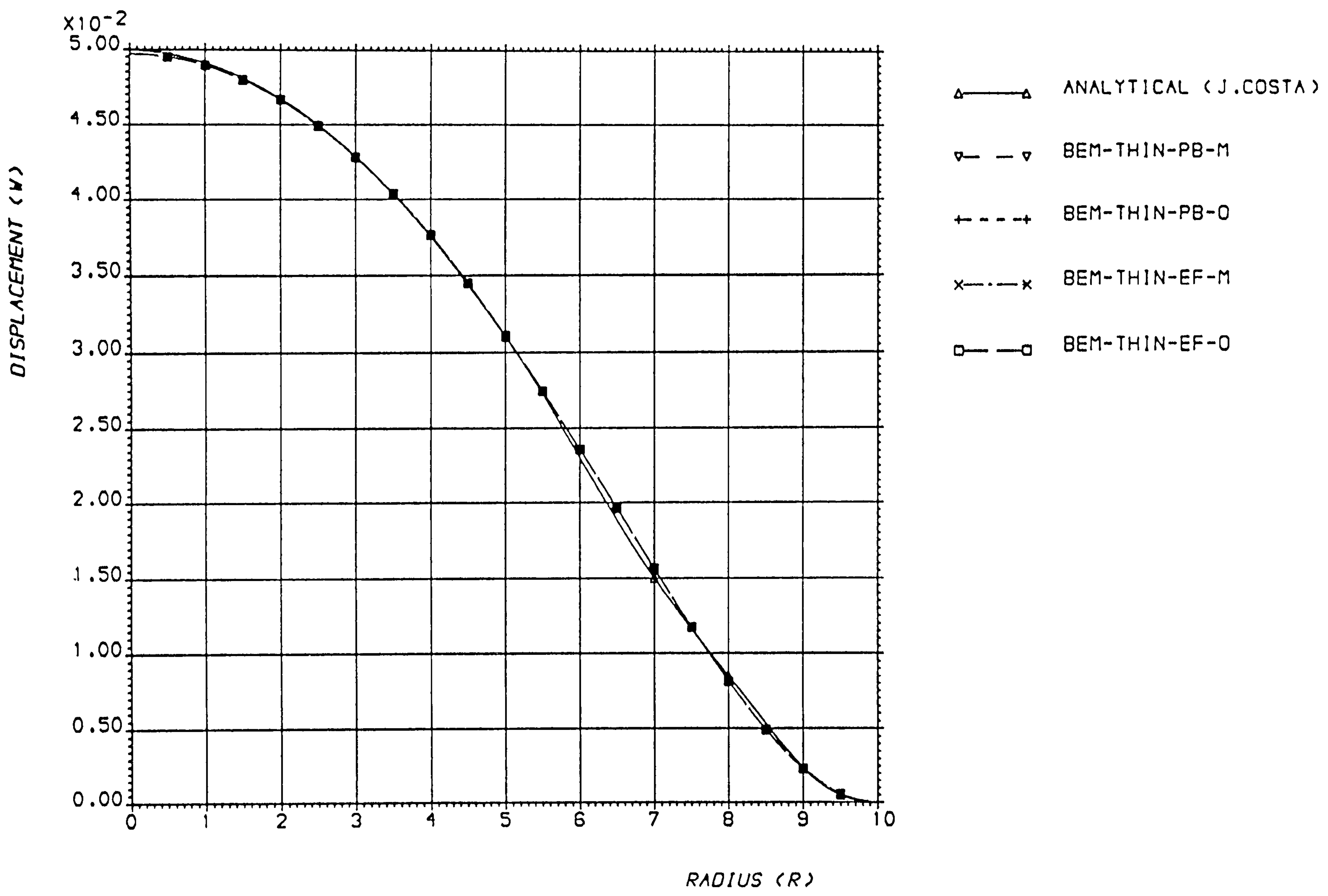


FIG. (7.4) DISPLACEMENT DISTRIBUTION OF CLAMPED CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER UNIFORM DOMAIN LOADING

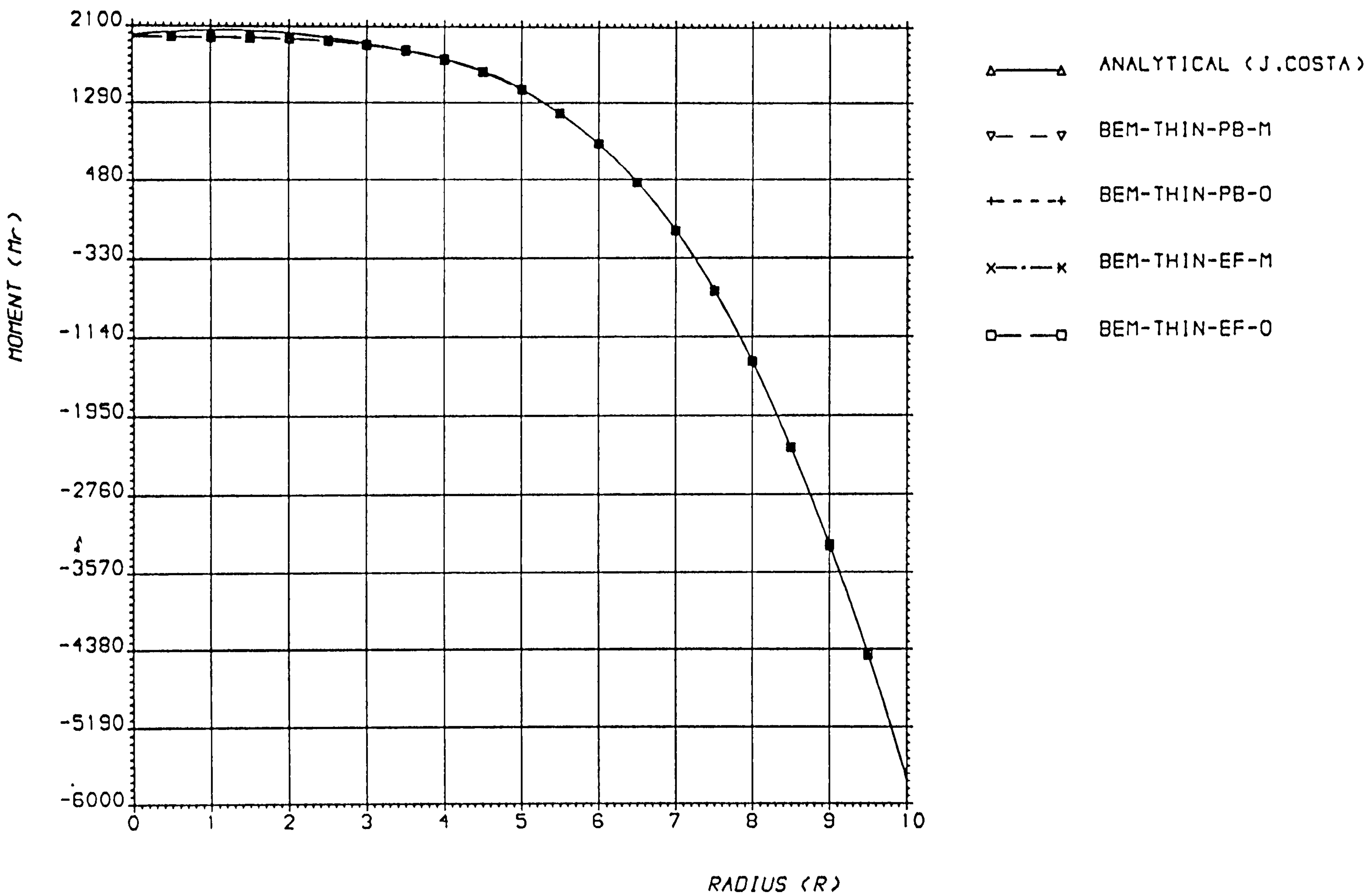


FIG. (7.5) DISTRIBUTION OF BENDING MOMENT (Mr) OF CLAMPED CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

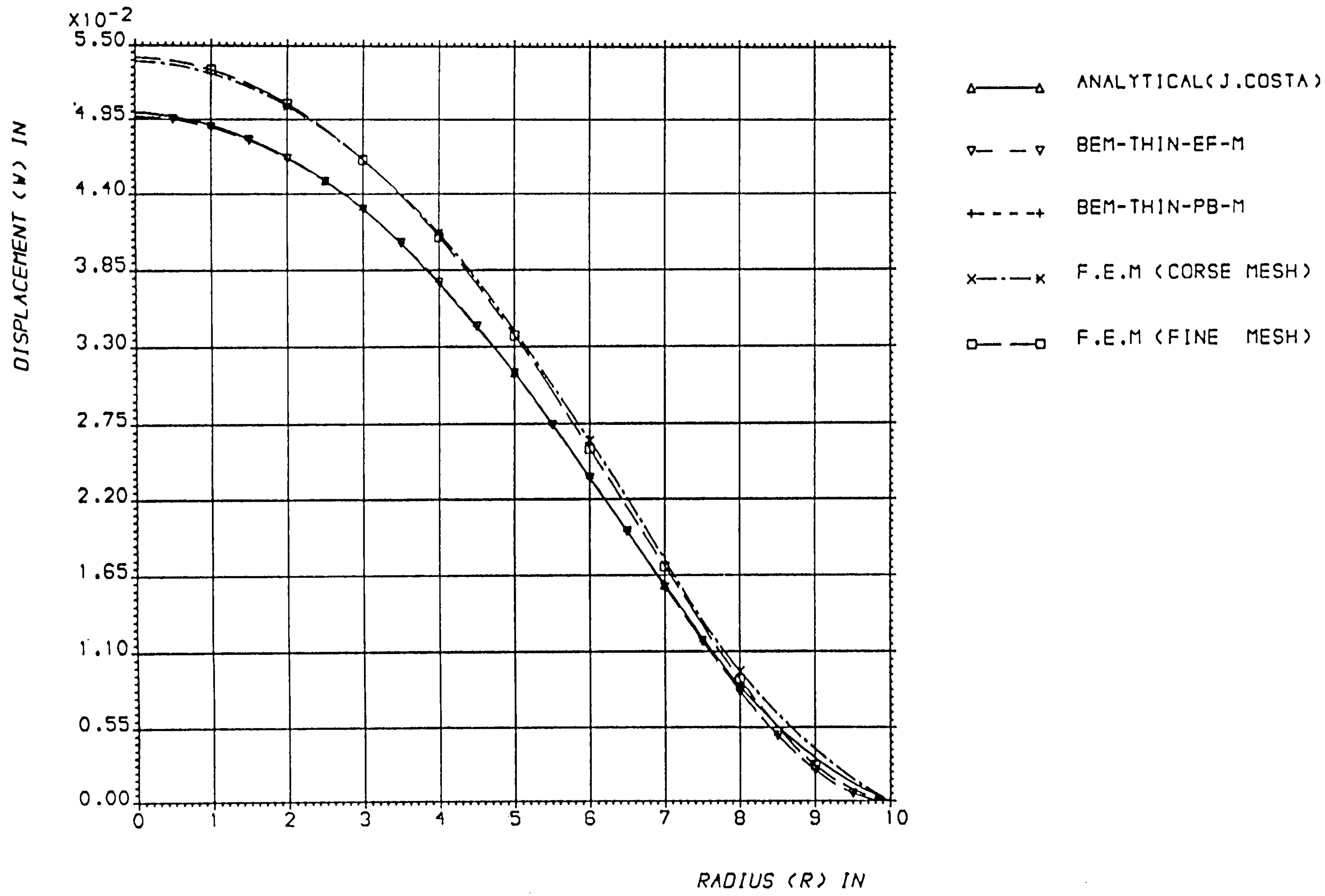


FIG. (7.6) FEM vs BEM DISPLACEMENT DISTRIBUTION OF CLAMPED CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMIAN LOADING.

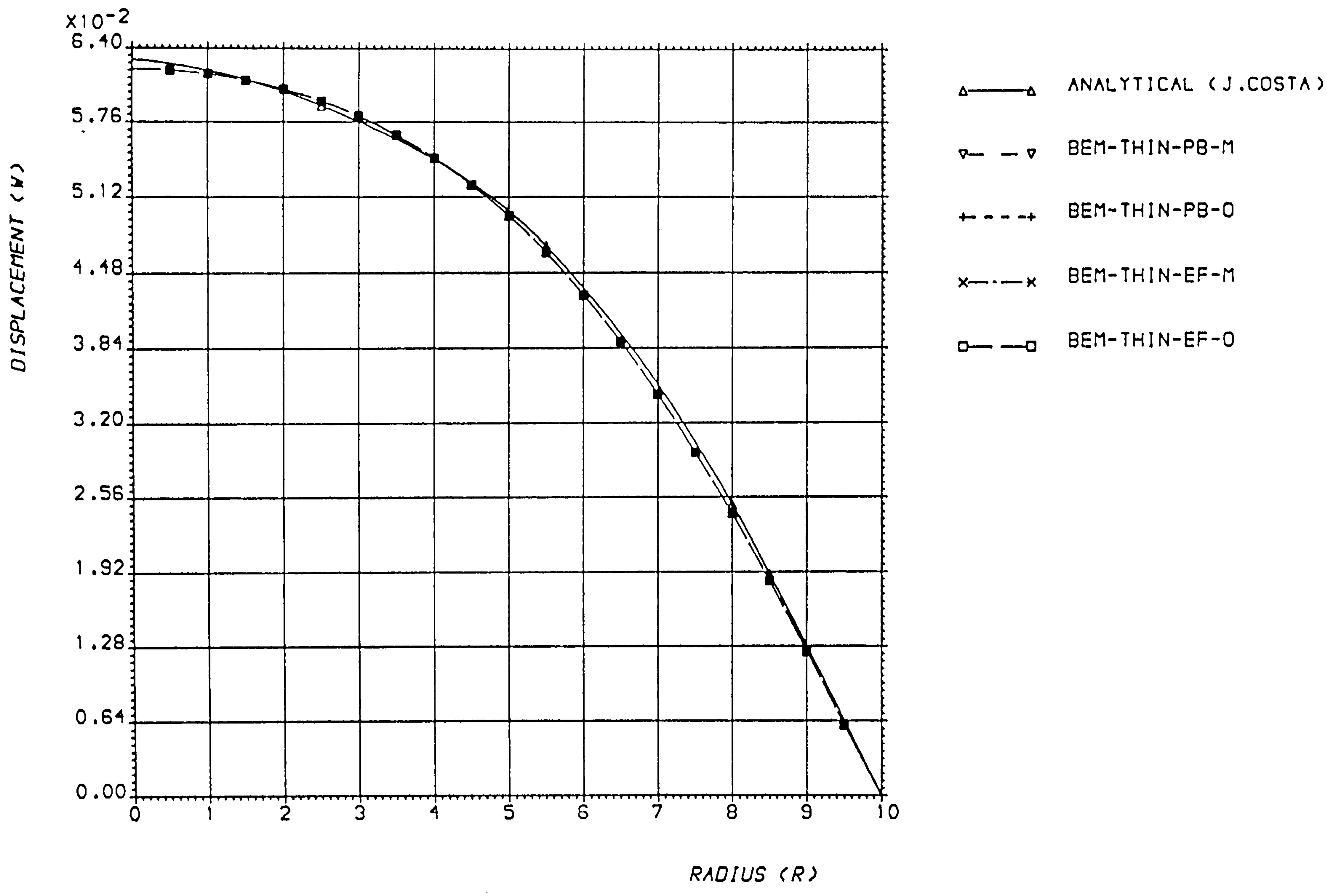


FIG. (7.7) DISPLACEMENT DISTRIBUTION OF SIMPLY-SUPPORTED CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER UNIFORM DOMAIN LOADING.

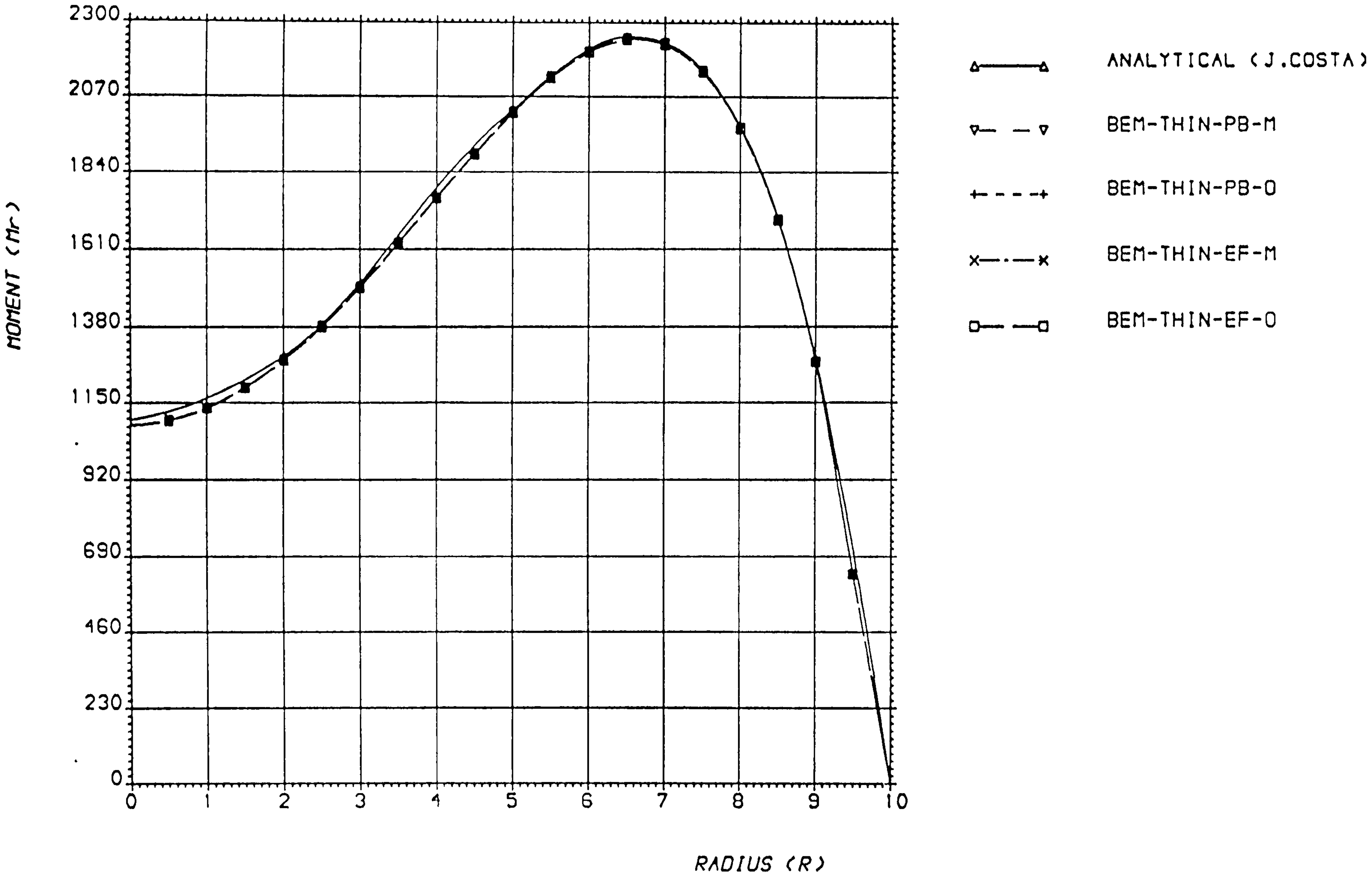


FIG. (7.8) BENDING MOMENT (Mr) DISTRIBUTION FOR SIMPLY-SUPPORTED CIRCULAR PLATE UNDER DOMAIN LOADING.

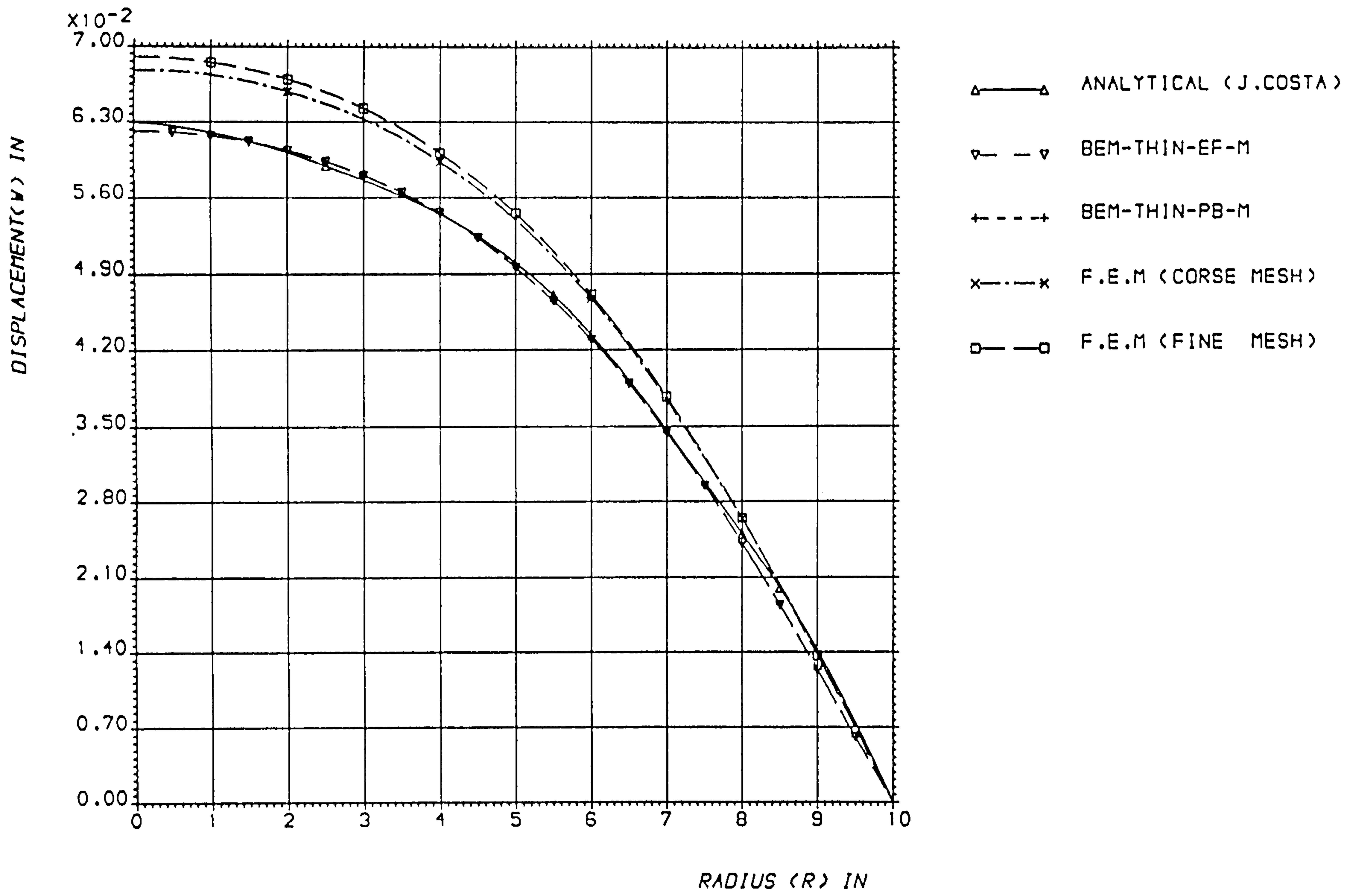


FIG. (7.9) FEM vs BEM DISPLACEMENT DISTRIBUTION OF SIMPLY-SUPPORTED CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

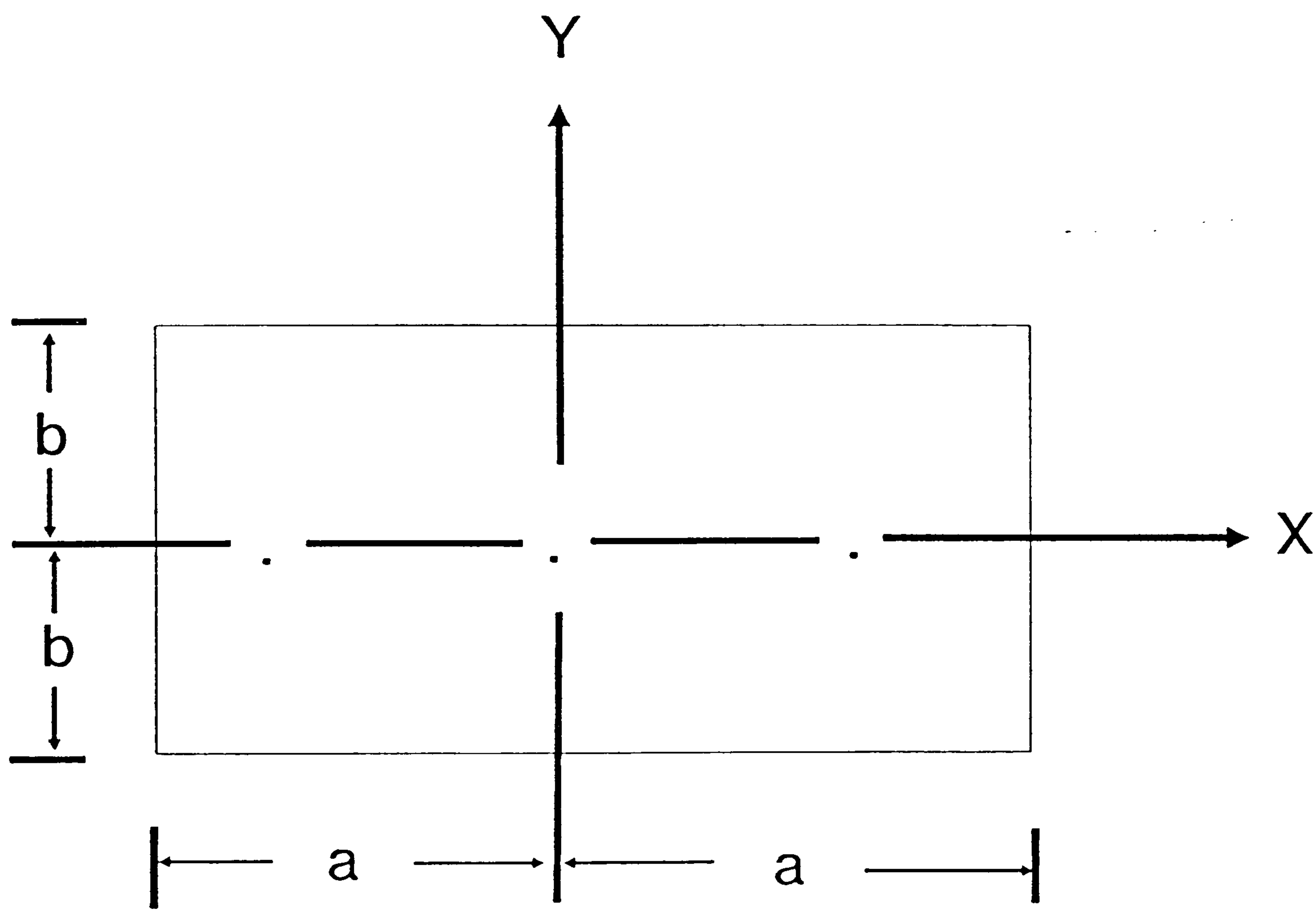


FIG [7.10] Clamped Rectangular Plate Example

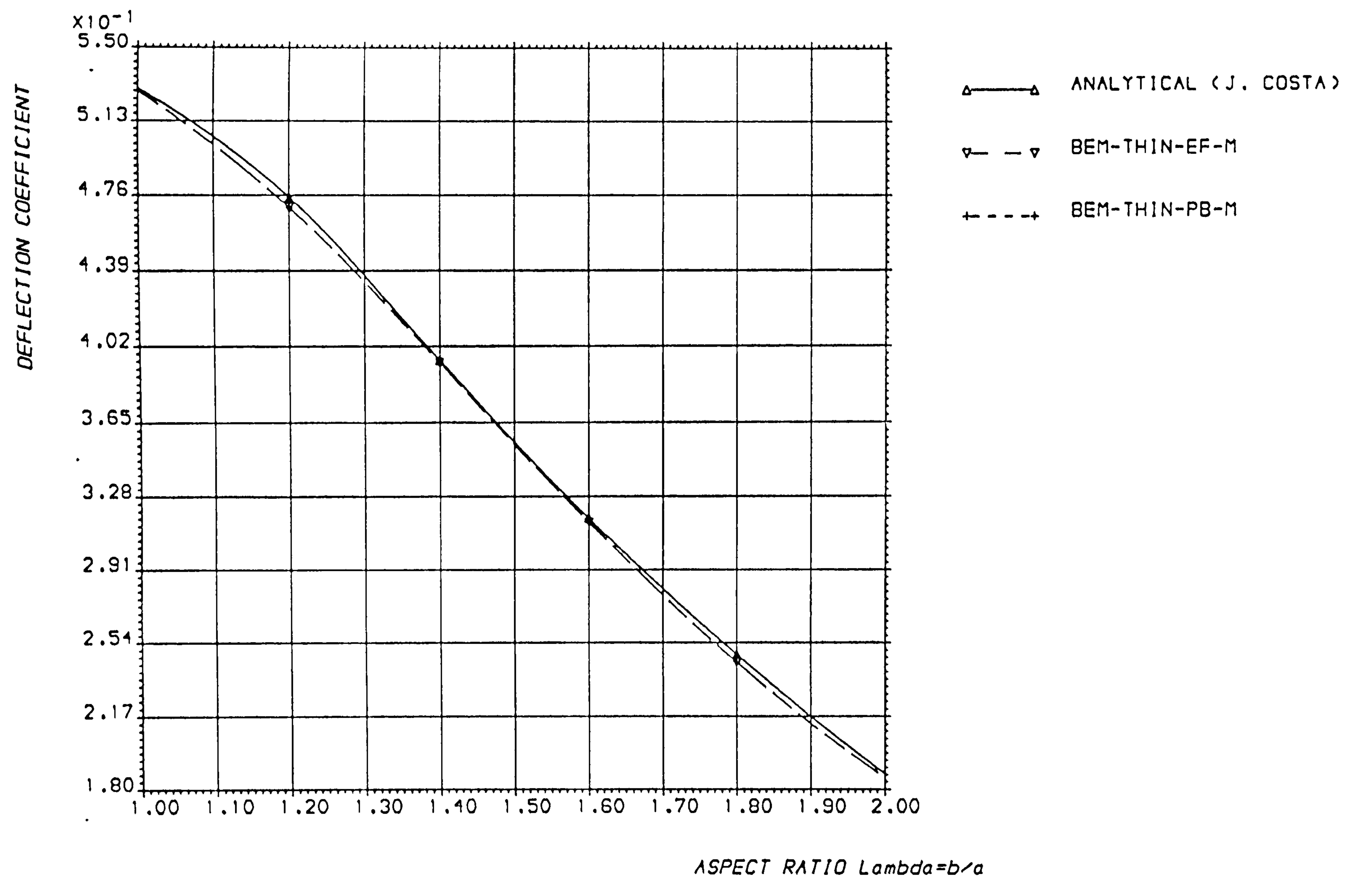


FIG. (7.11) VARIATION OF CENTRE DEFLECTION WITH ASPECT RATIO FOR CLAMPED RECTANGULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING

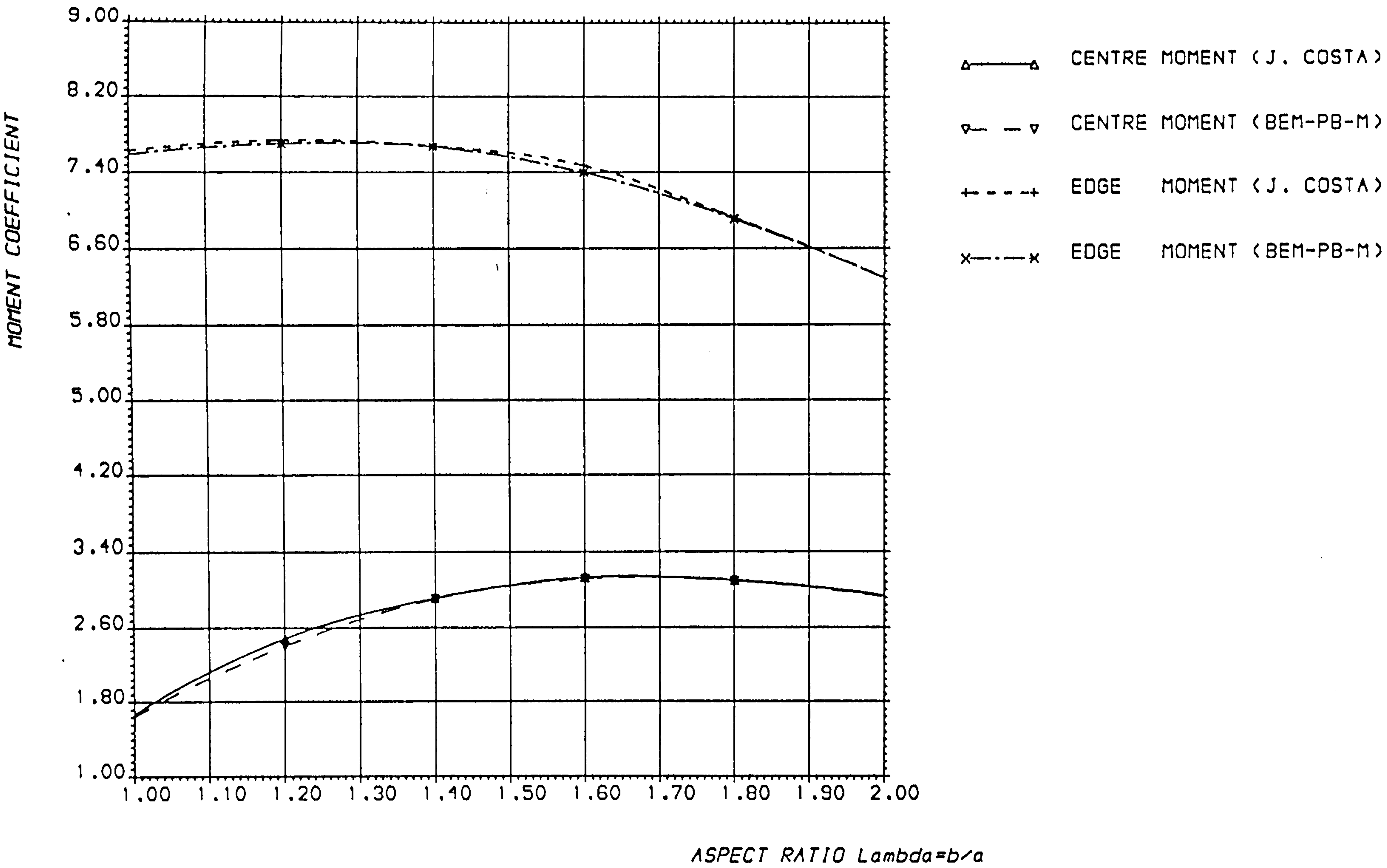


FIG. (7.12) VARIATION OF CENTRE AND MAXIMUM EDGE MOMENTS WITH ASPECT RATIO FOR CLAMPED RECTANGULAR PLATE UNDER DOMAIN LOADING.

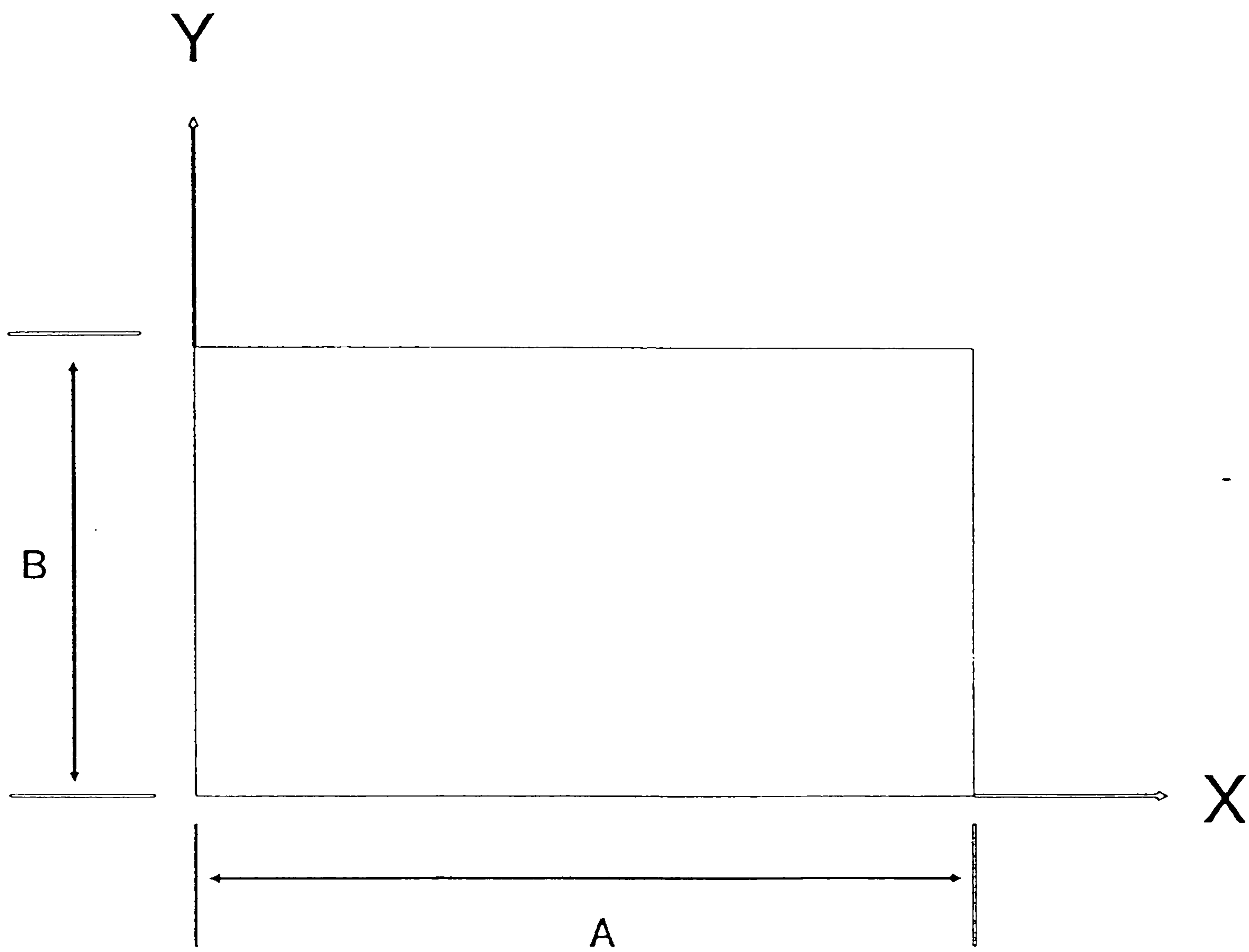


FIG [7.13] Rectangular plate used for simply-supported case.

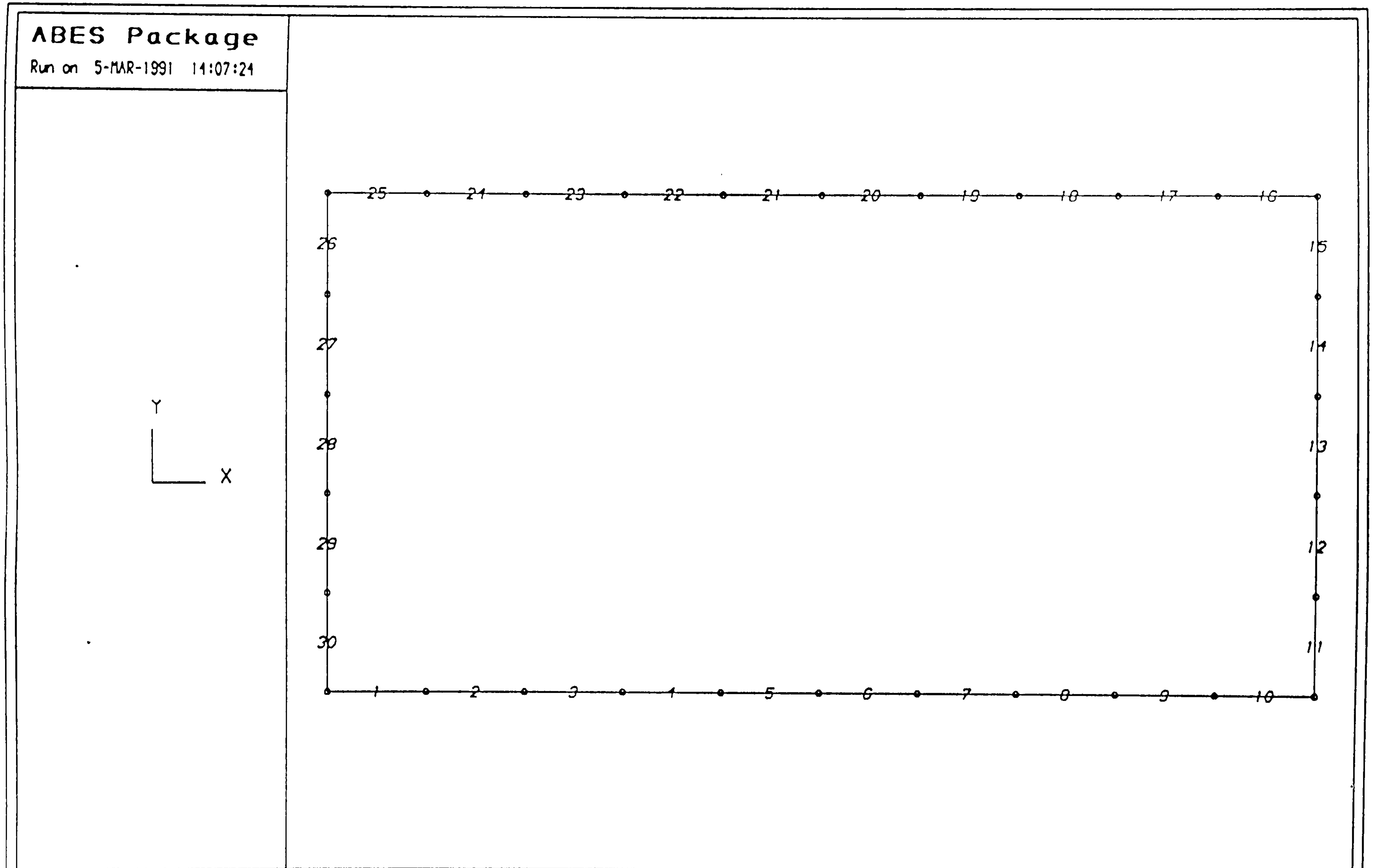


FIG [7.14] BOUNDARY ELEMENT MESH FOR RECTANGULAR PLATE CASE STUDIES

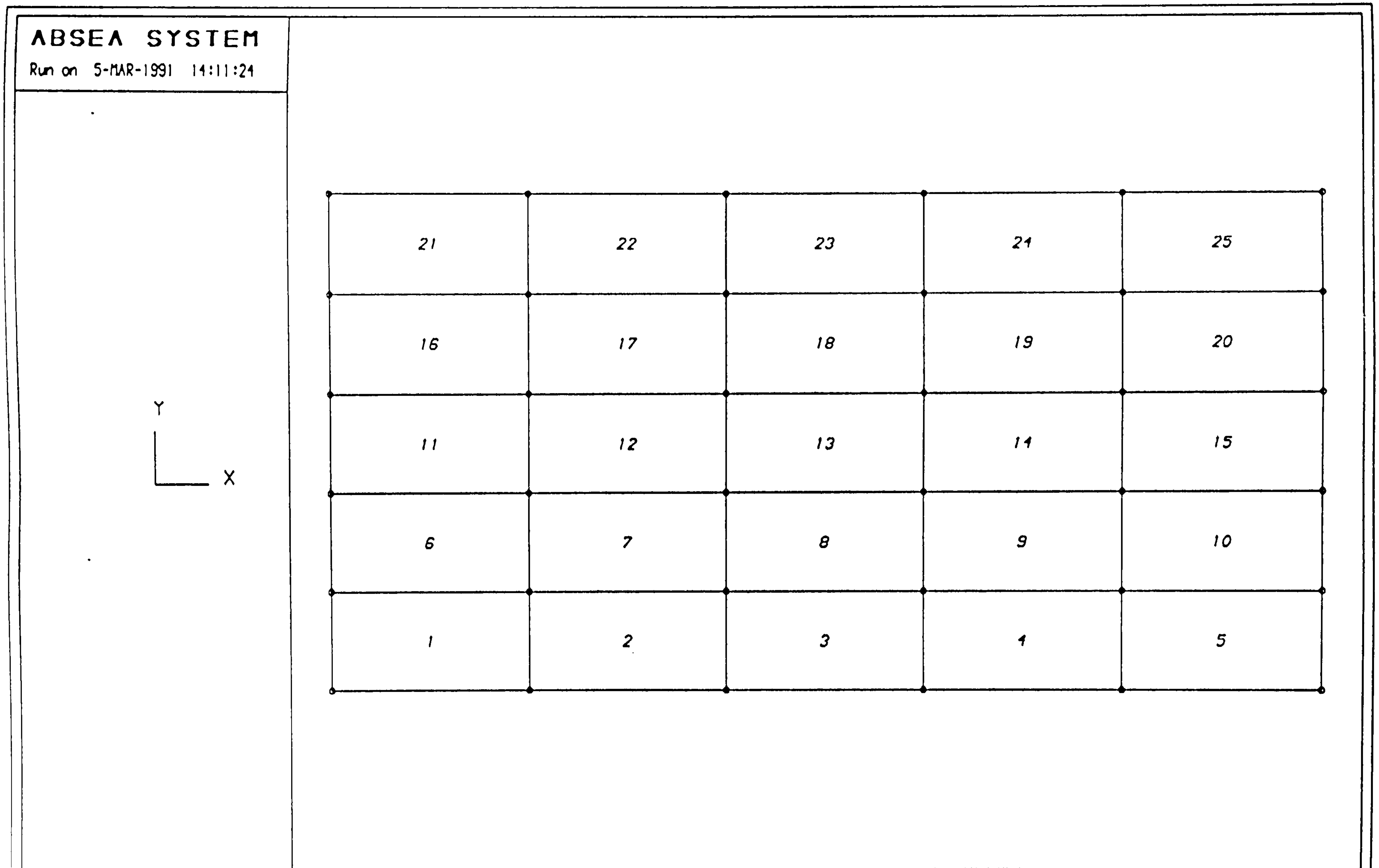


FIG [7.15] COARSE FINITE ELEMENT MESH FOR RECTANGULAR PLATE CASE STUDIES

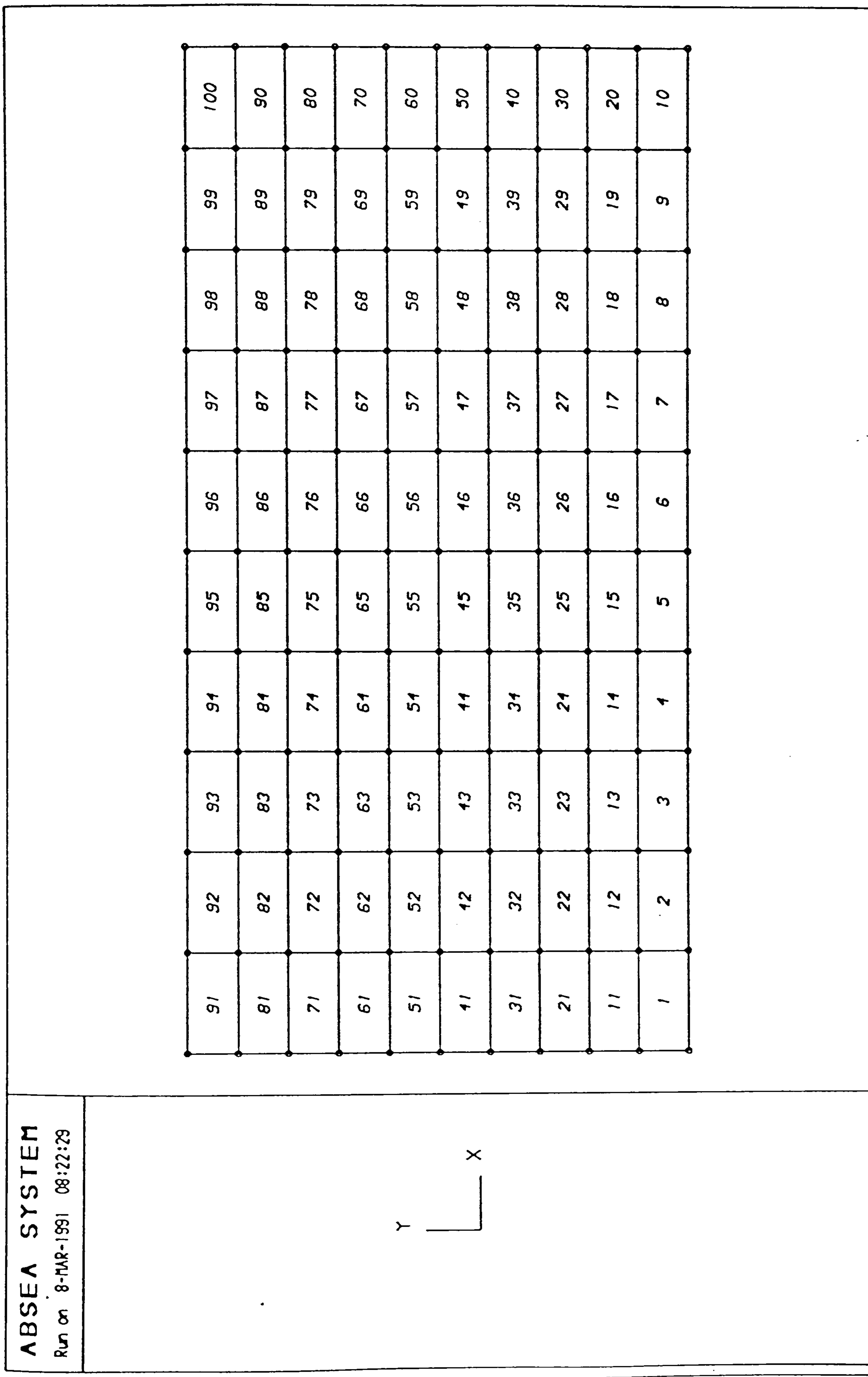


FIG [7.16] FINE FINITE ELEMENT MESH FOR RECTANGULAR PLATE CASE STUDIES

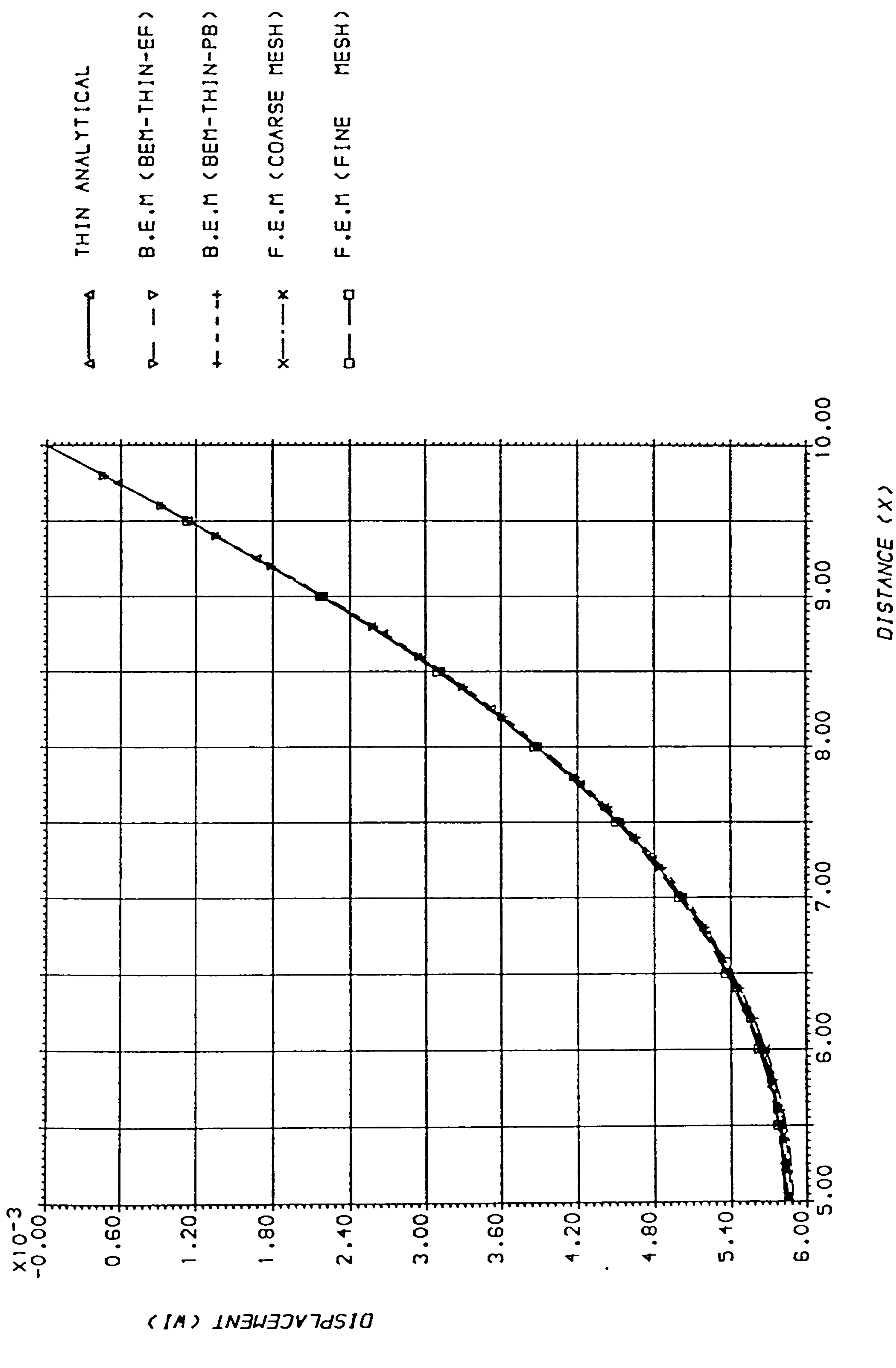


FIG (7.17) DISPLACEMENT FOR THIN SIMPLY-SUPPORTED RECTANGULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

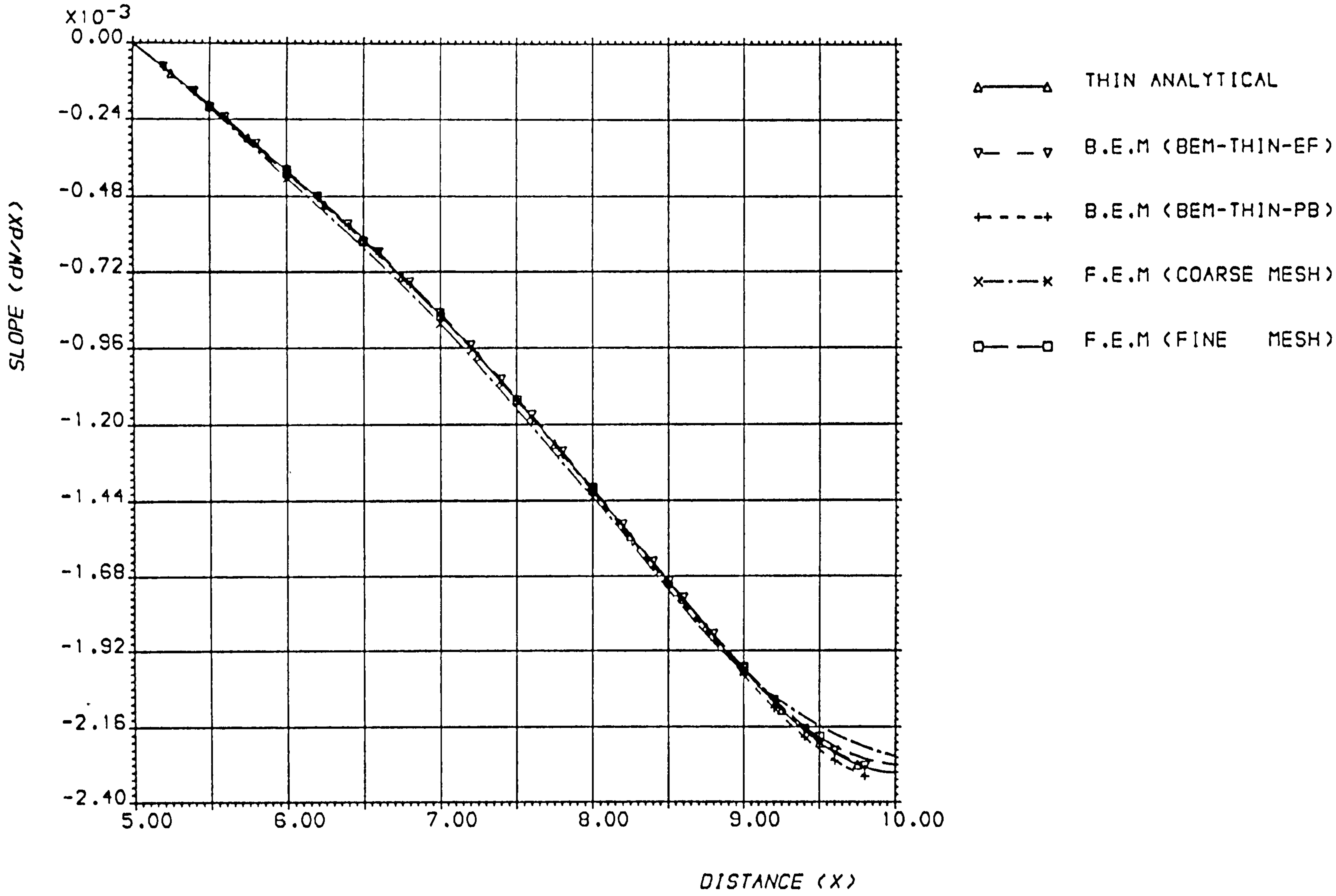


FIG (7.18) SLOPE (dW/dX) FOR THIN SIMPLY-SUPPORTED RECTANGULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

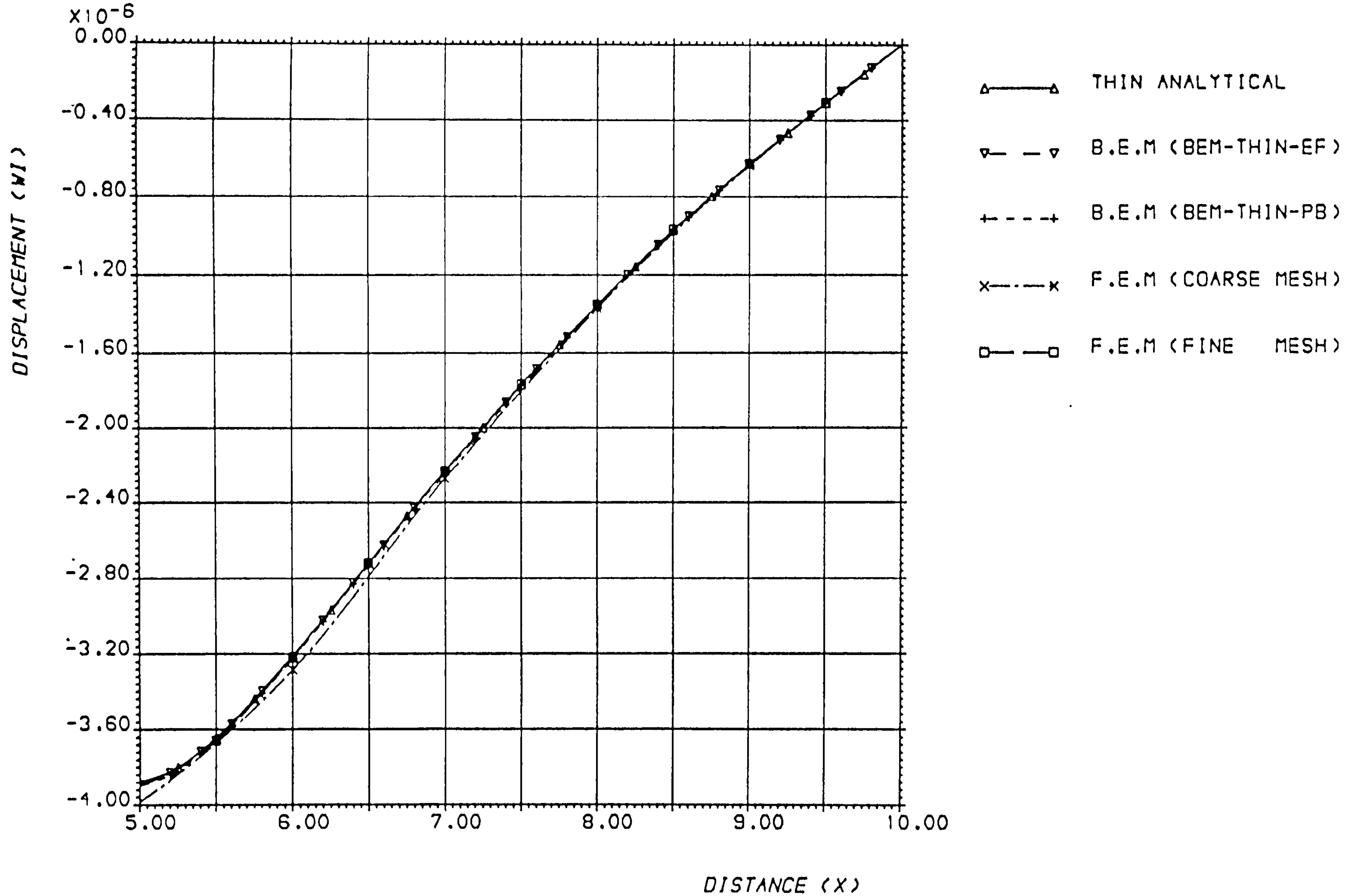


FIG (7.19) DISPLACEMENT DISTRIBUTION FOR THIN SIMPLY-SUP RECTANGULAR PLATE ON ELASTIC FOUNDATION UNDER CONCENTRATED LOADING

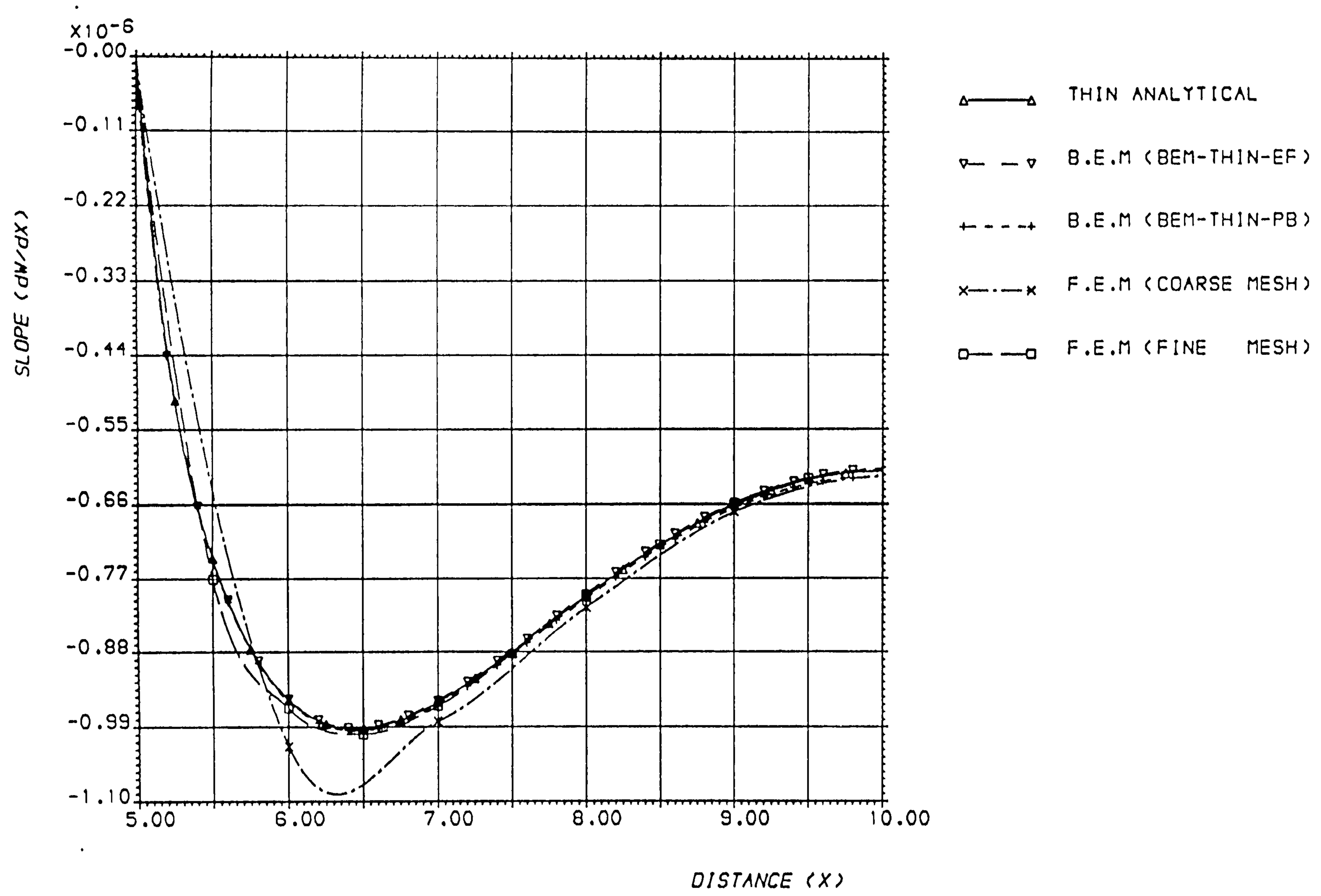


FIG (7.20) SLOPE (dW/dX) FOR THIN SIMPLY-SUPPORTED RECTANGULAR PLATE ON ELASTIC FOUNDATION UNDER CONCENTRATED LOADING.

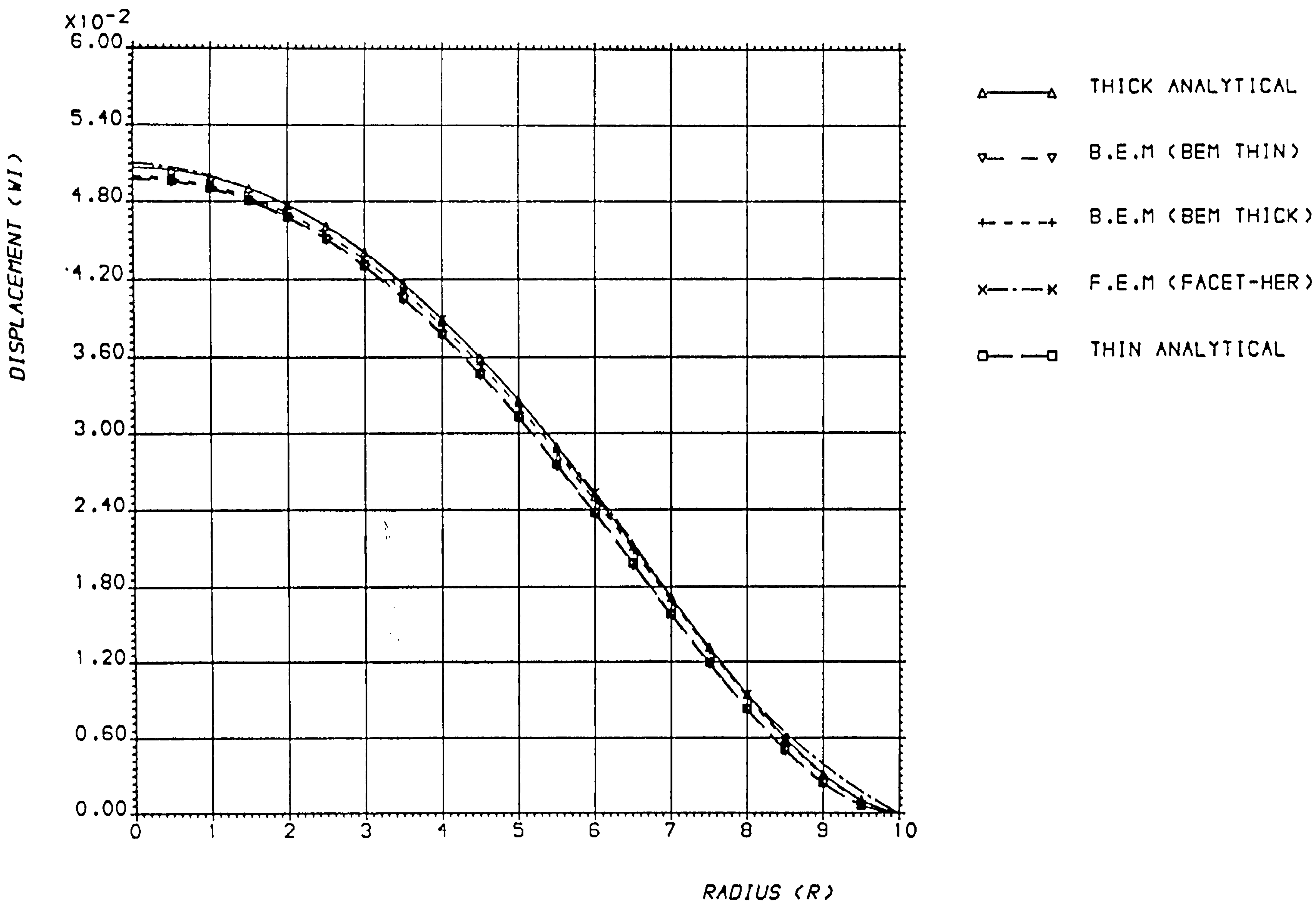


FIG (7.21) DISPLACEMENT DISTRIBUTION FOR CLAMPED THIN (h=1) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

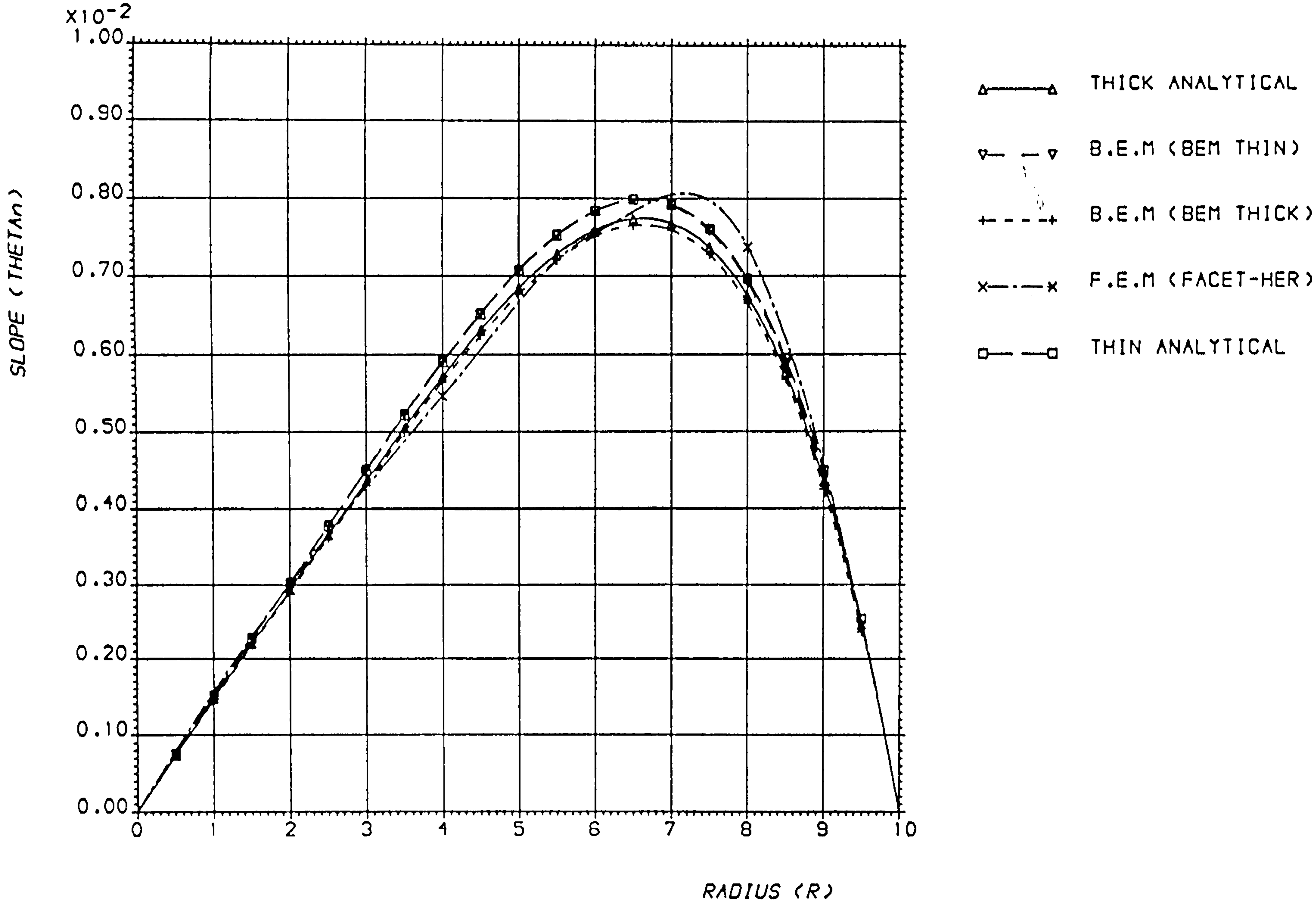


FIG (7.22) SLOPE DISTRIBUTION FOR CLAMPED THIN (h=1) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

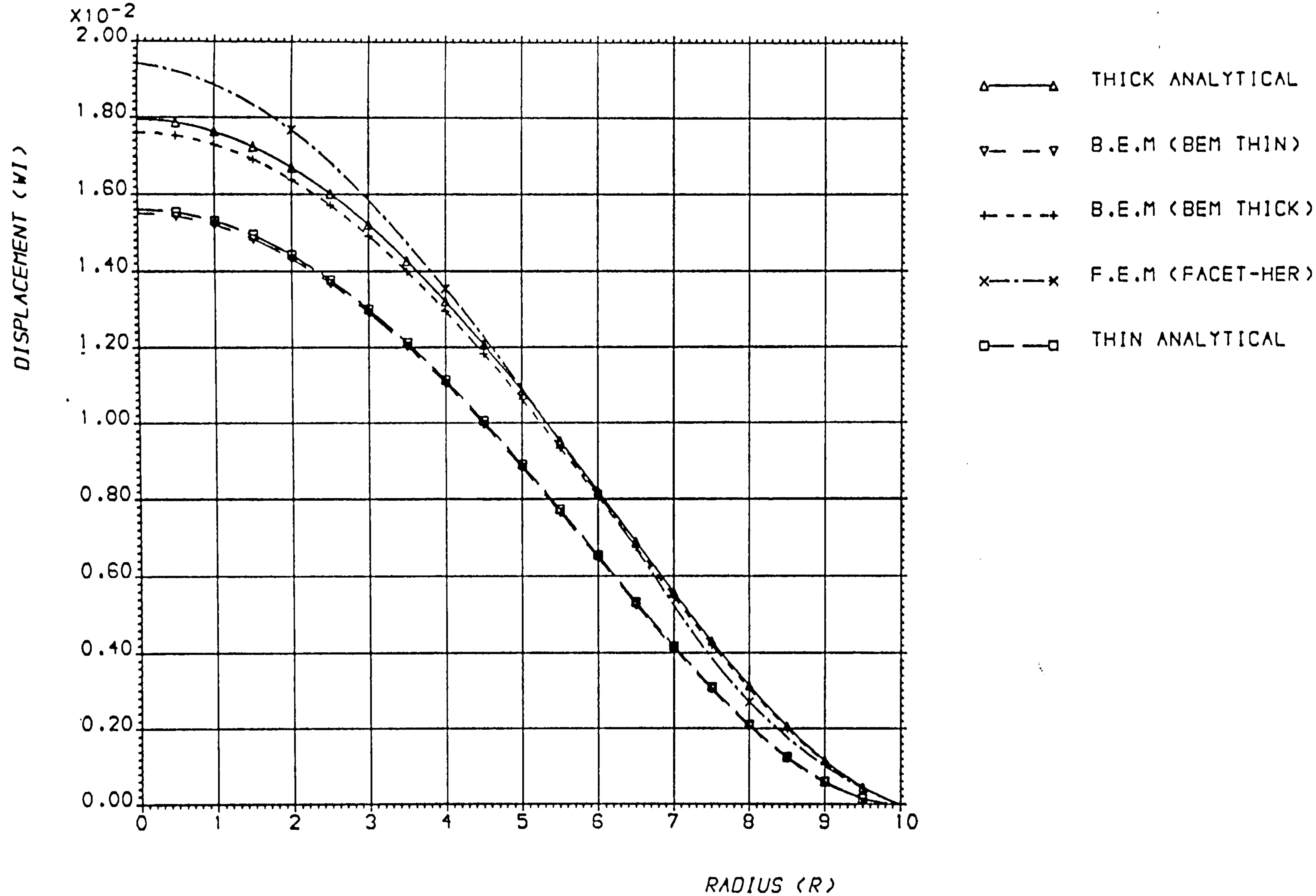


FIG (7.23) DISPLACEMENT DISTRIBUTION FOR CLAMPED THICK (h=2) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

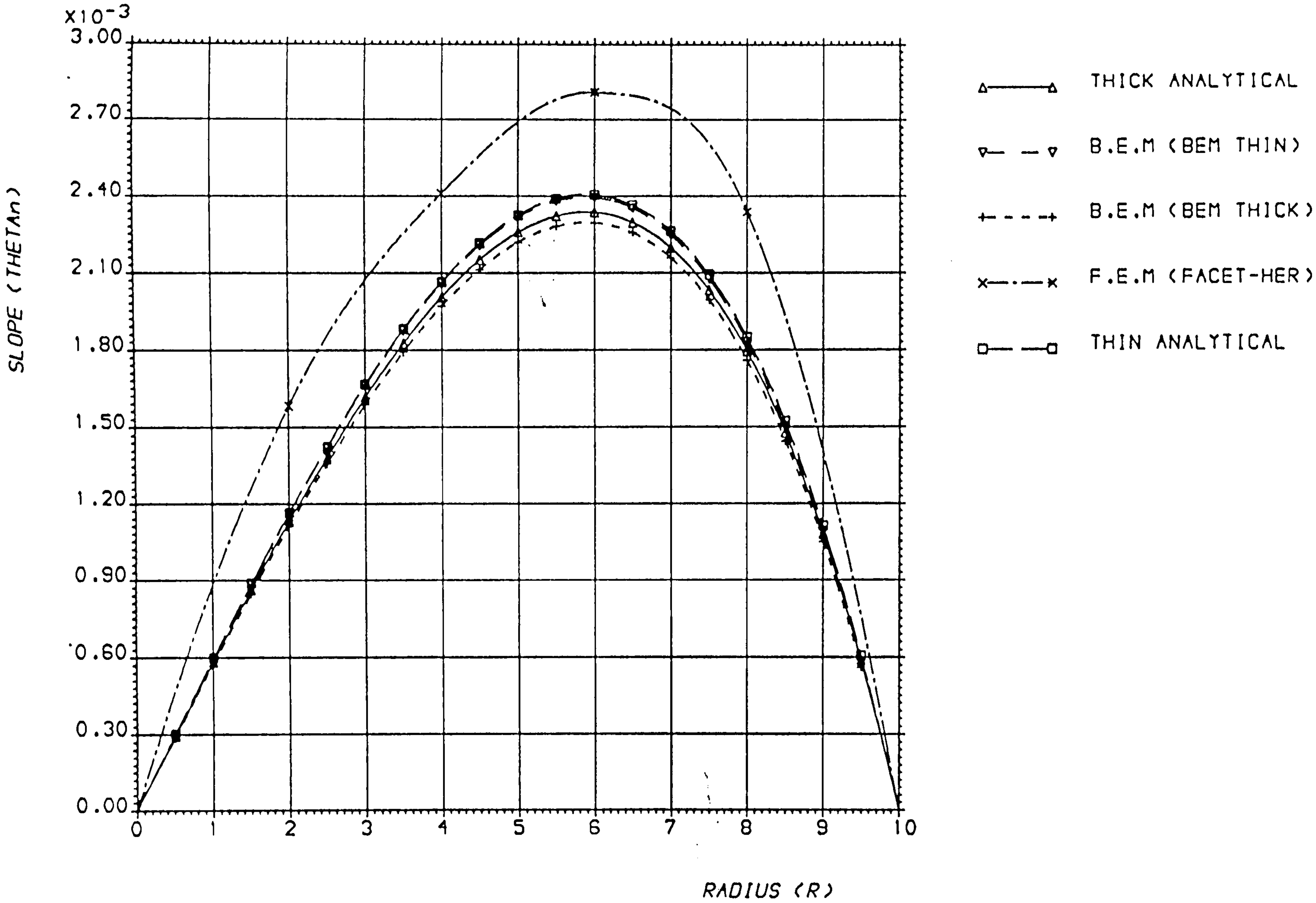


FIG (24) SLOPE DISTRIBUTION FOR CLAMPED THICK (h=2) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

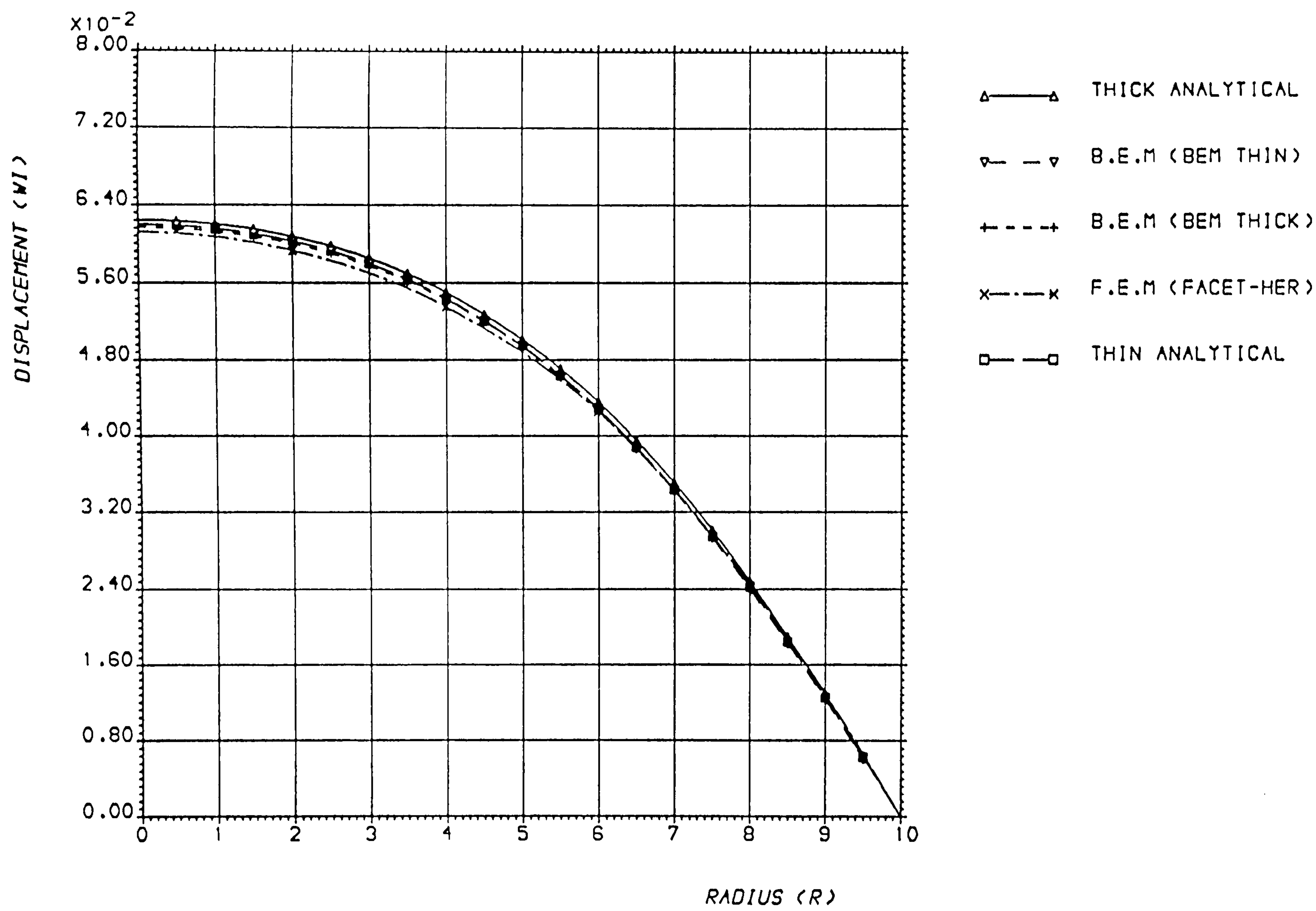


FIG (7.25) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUPPORTED THIN (h=1) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

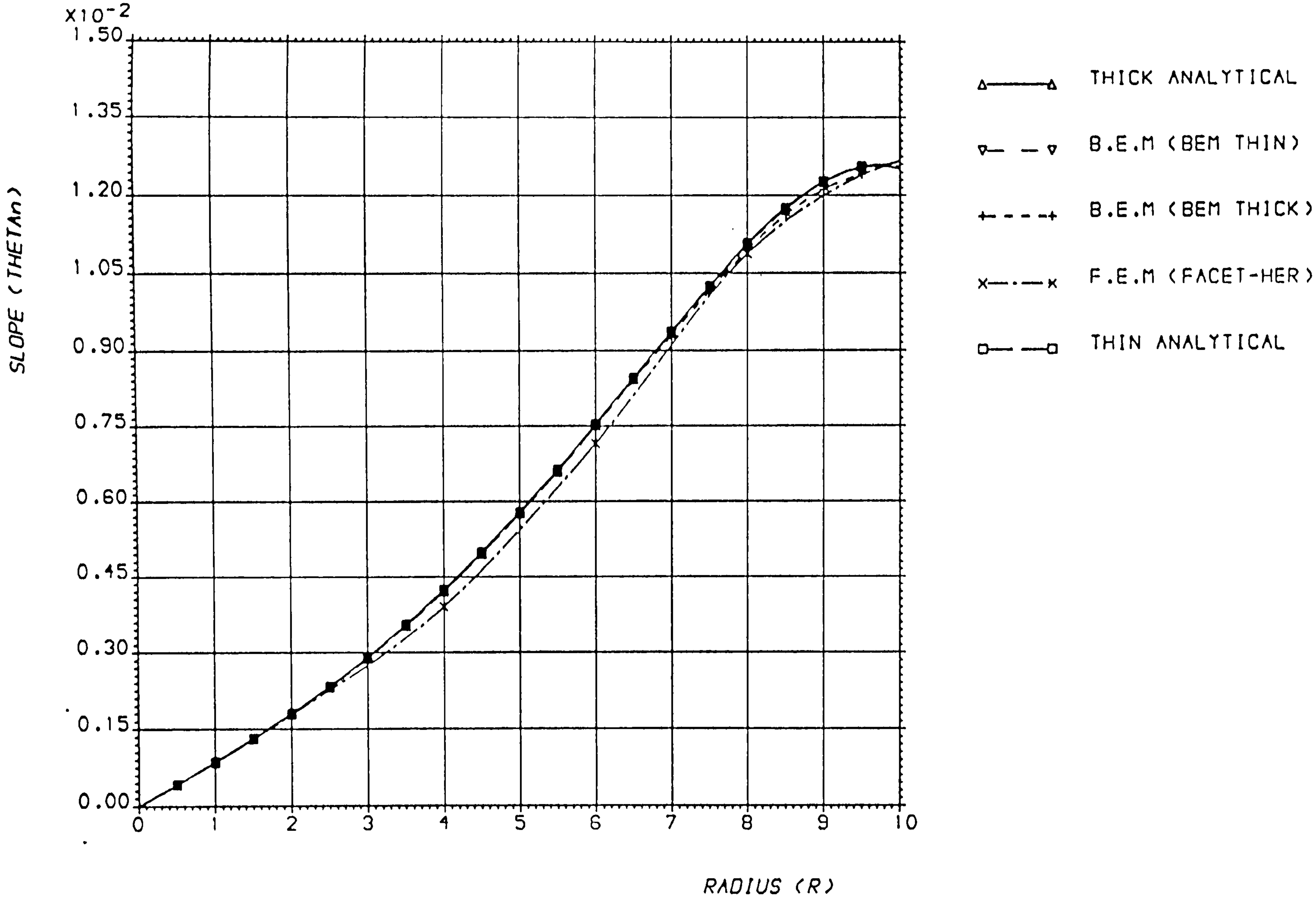


FIG (7.26) SLOPE DISTRIBUTION FOR SIMPLY-SUPP. THIN (h=1) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

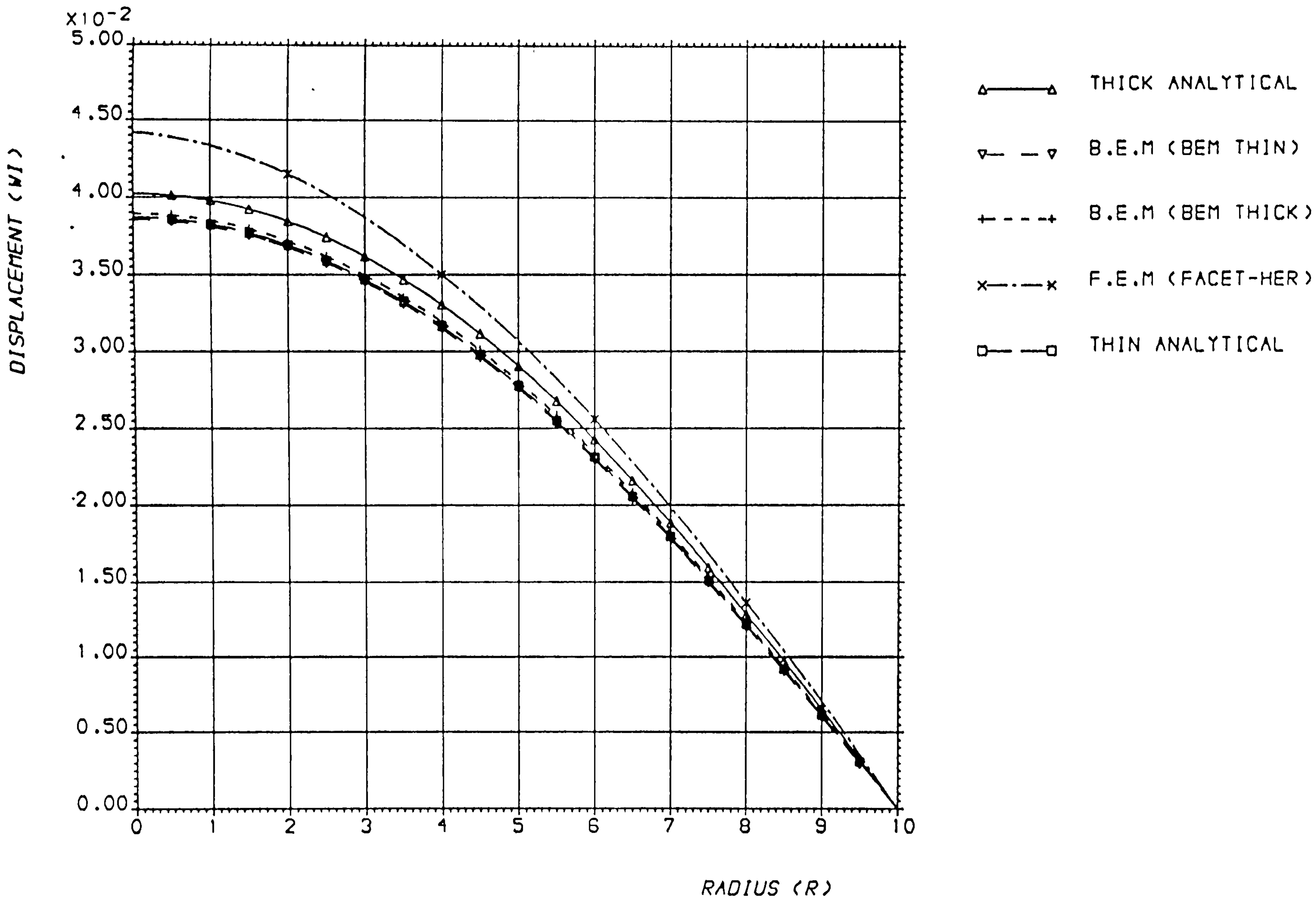


FIG (7.27) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUPPORTED THICK (h=2) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

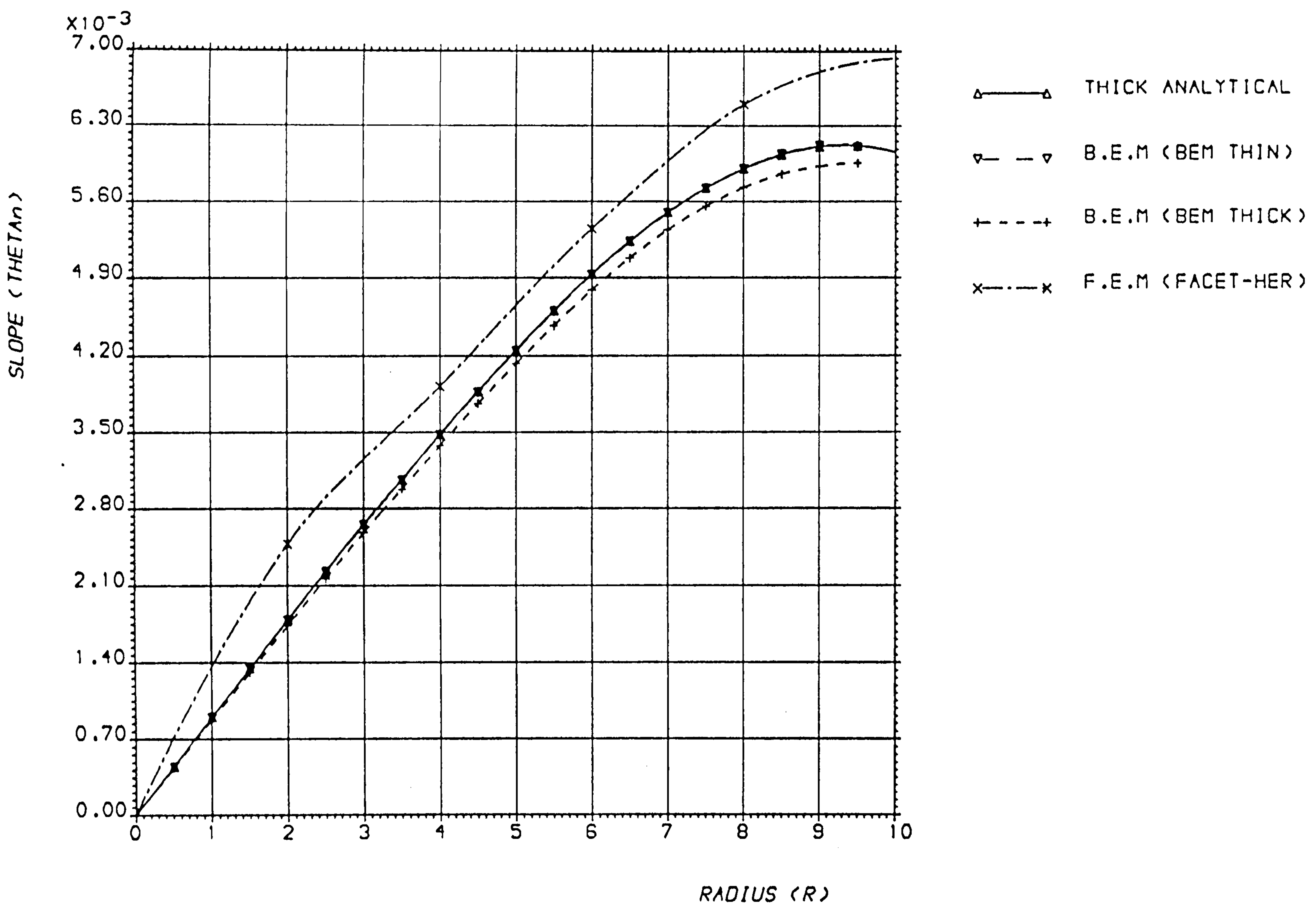


FIG (7.28) SLOPE DISTRIBUTION FOR SIMPLY-SUPPORTED THICK (h=2) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

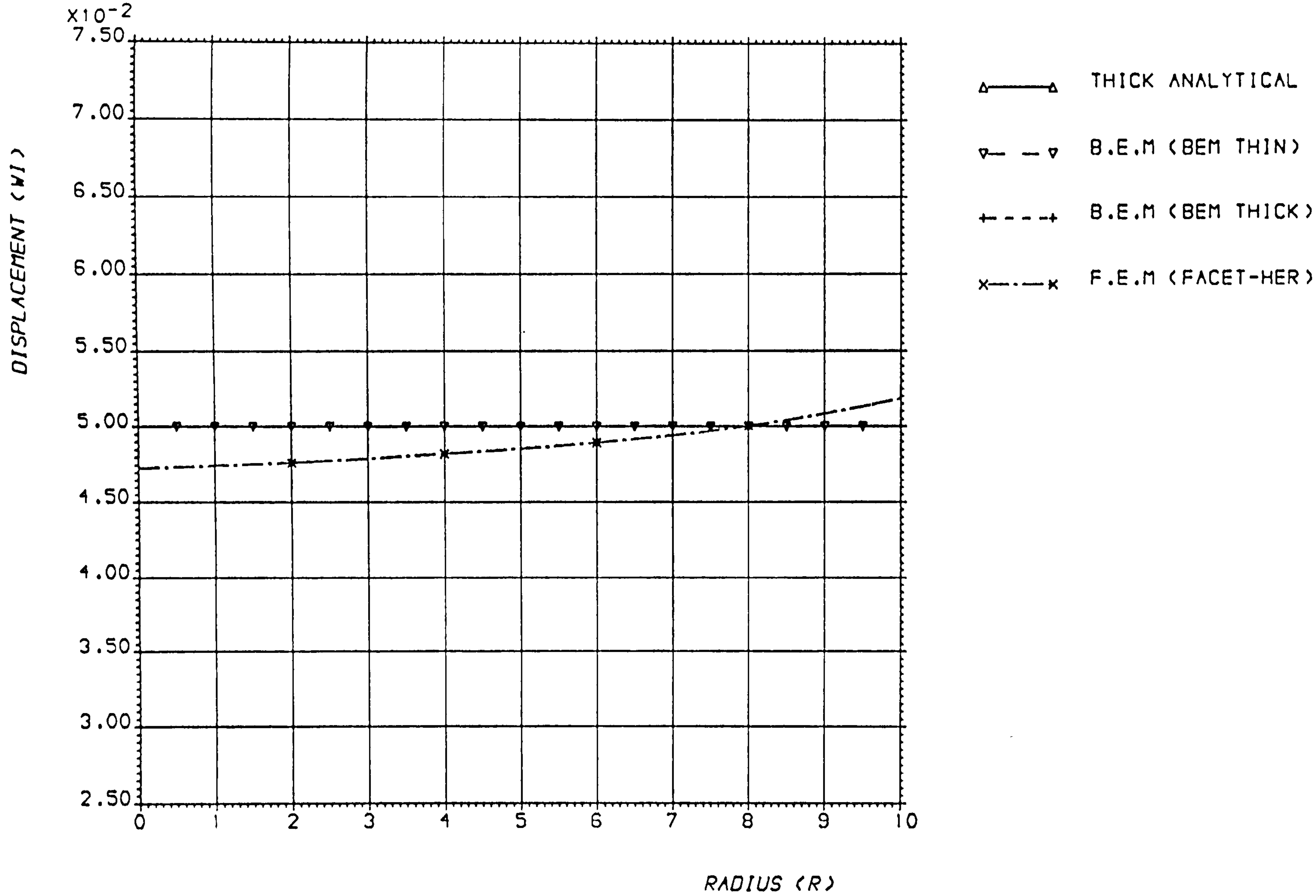


FIG (7.29) DISPLACEMENT DISTRIBUTION FOR FREE THIN CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

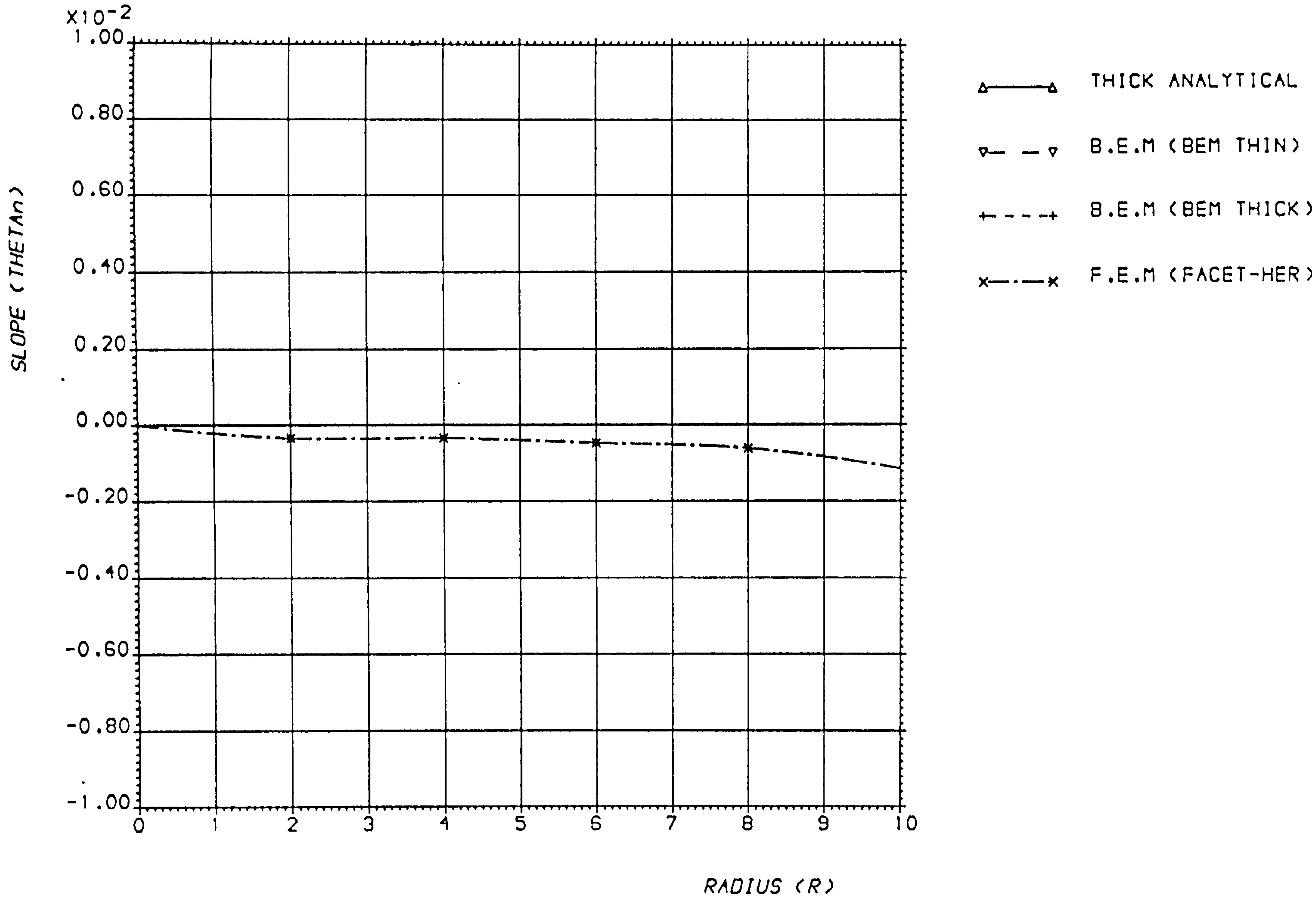


FIG (7.30) SLOPE DISTRIBUTION FOR FREE-FREE THIN CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER DOMAIN LOADING.

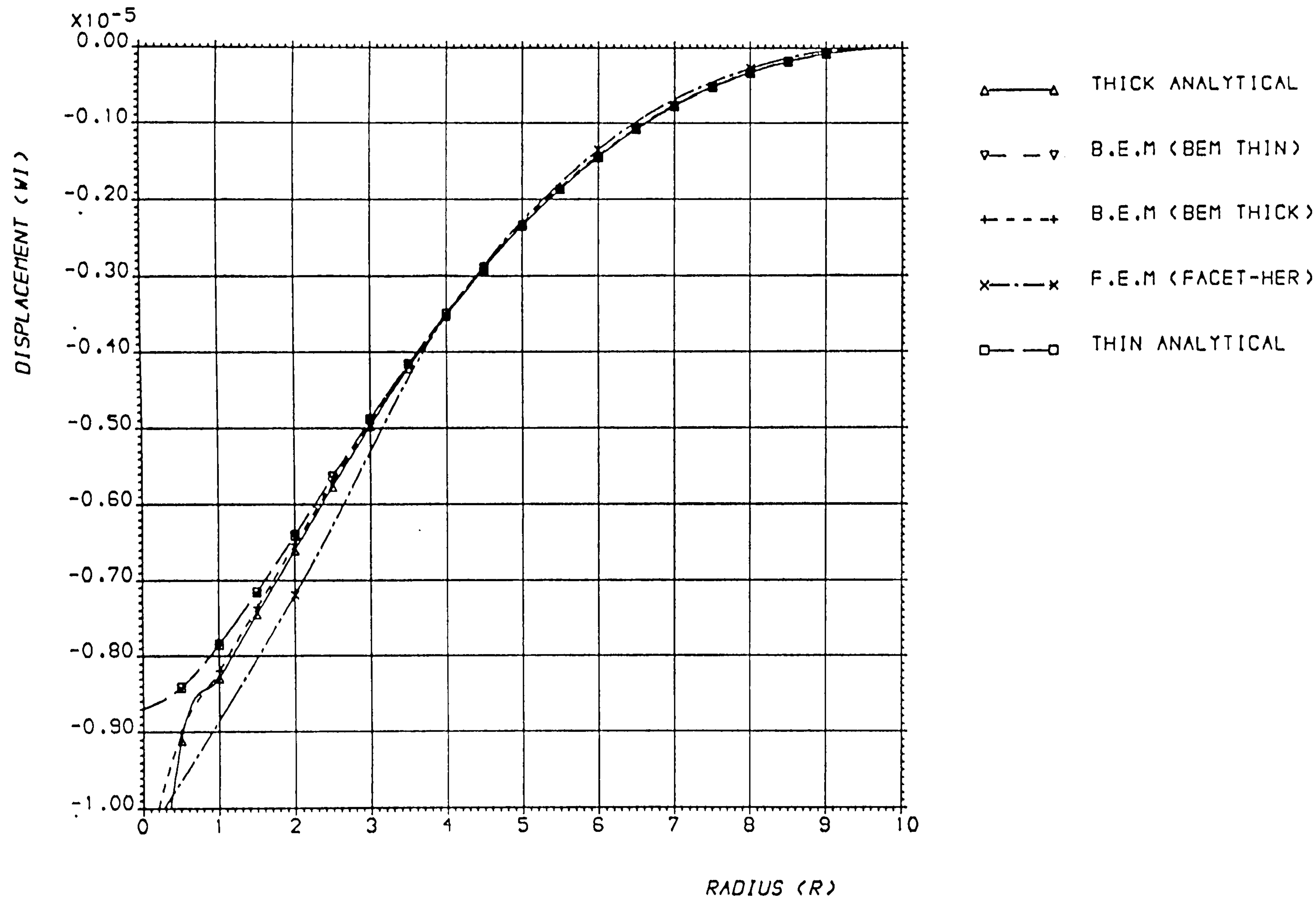


FIG (7.31) DISPLACEMENT DISTRIBUTION FOR CLAMPED THIN (h=1) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER CONCENTRATED LOADING.

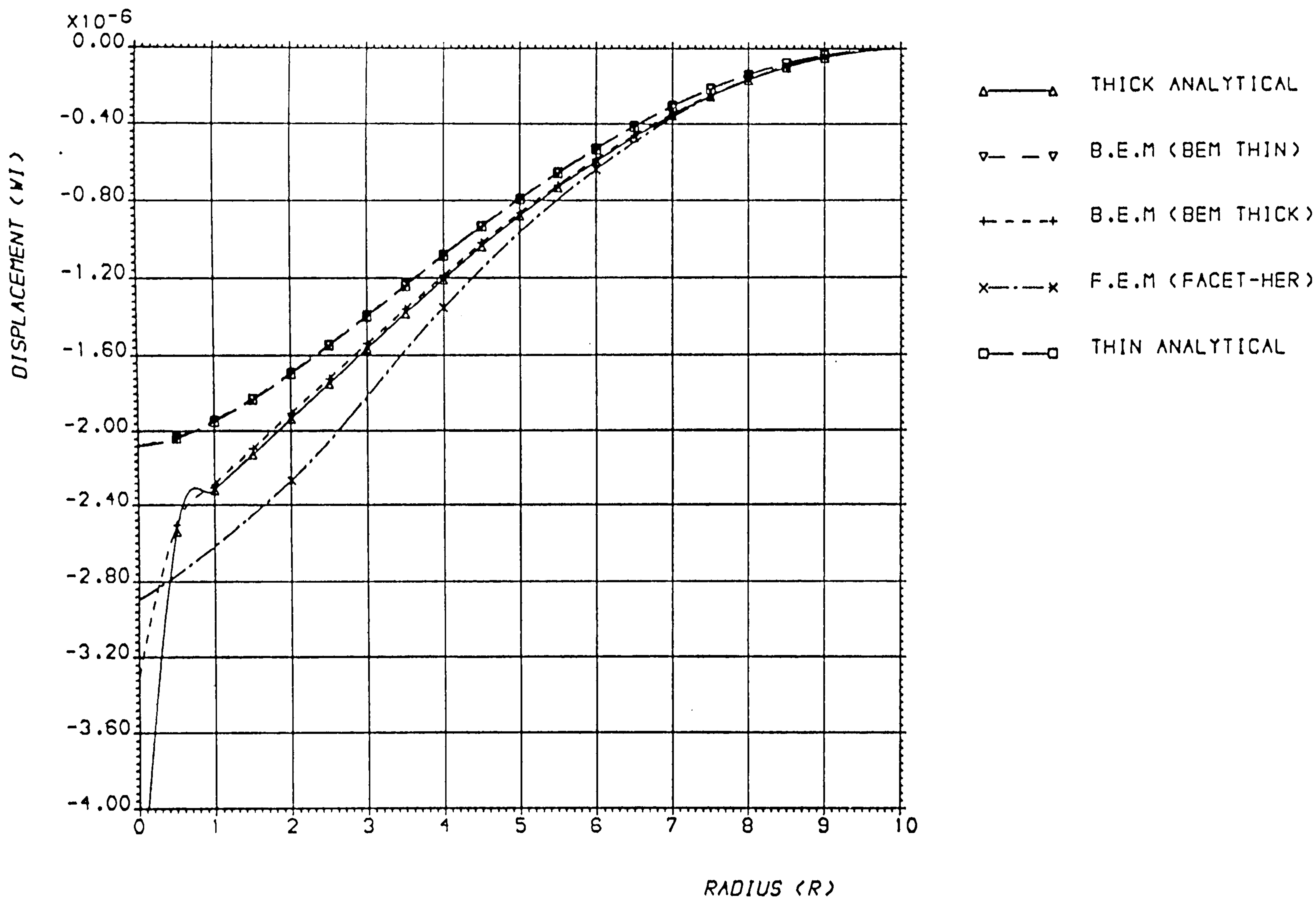


FIG (7.32) DISPLACEMENT DISTRIBUTION FOR CLAMPED THICK (h=2) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER CONCENTRATED LOADING.

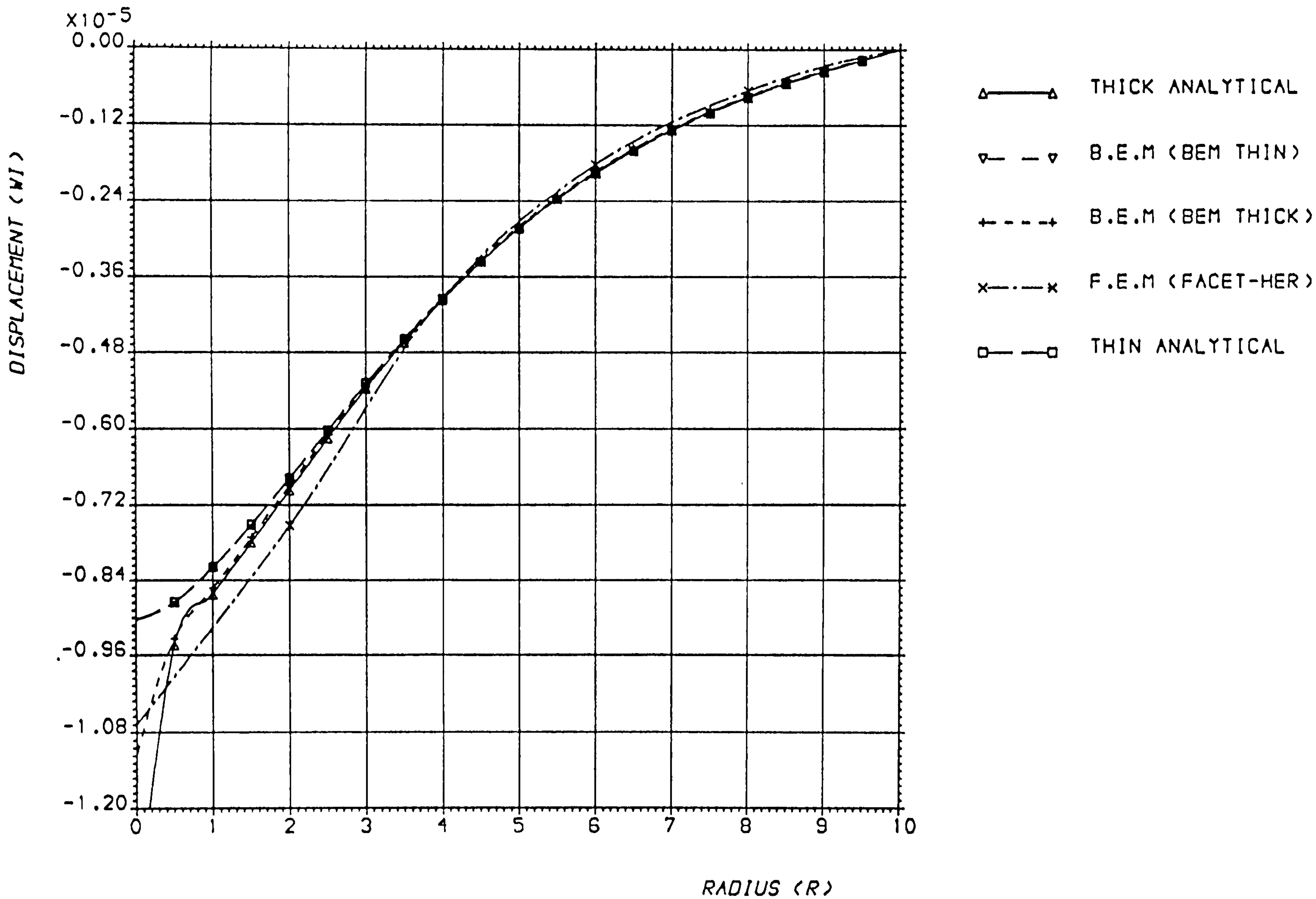


FIG (7.33) DISPLACEMENT FOR SIMPLY-SUPPORTED THIN (h=1) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER CONCENTRATED LOADING.

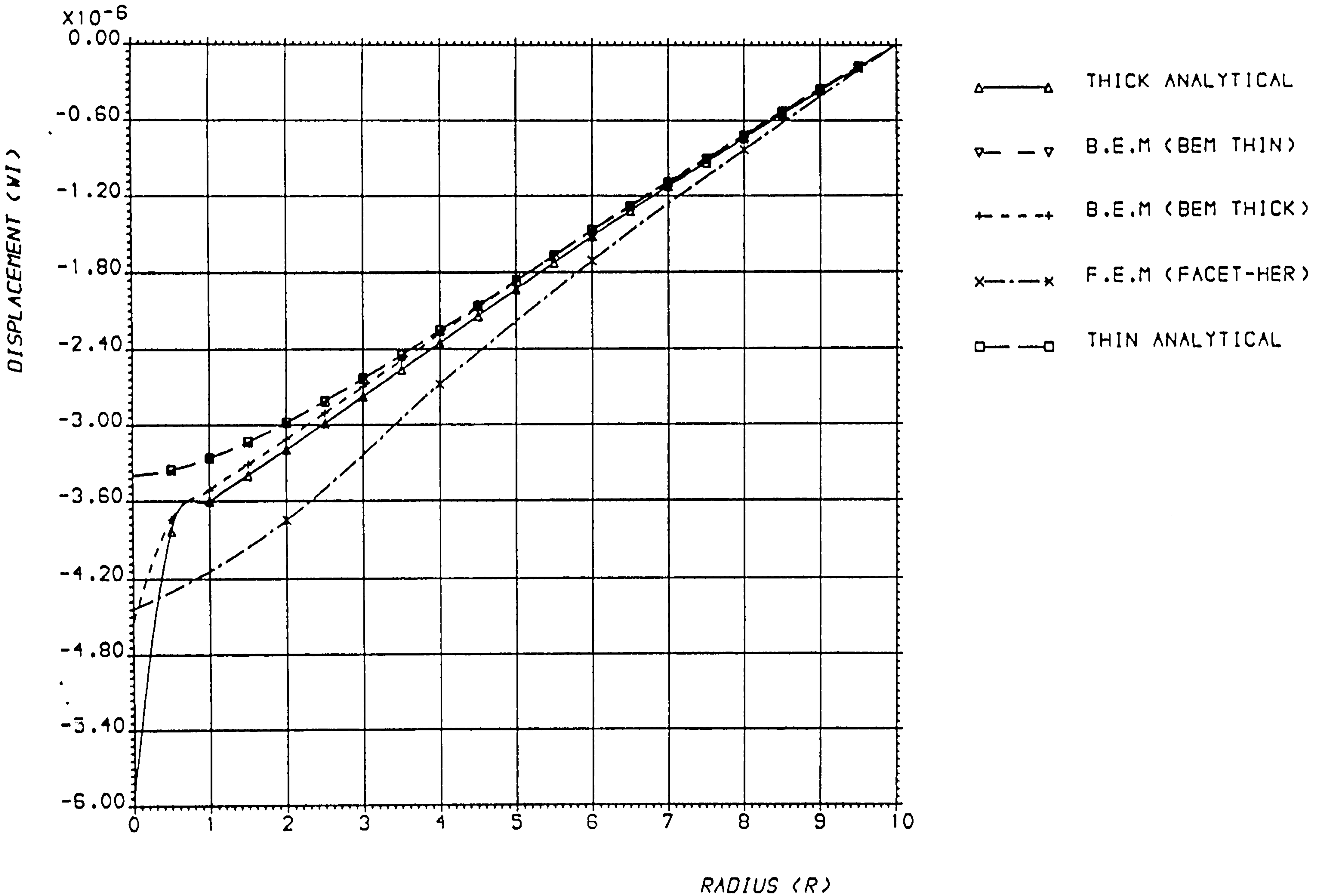


FIG (7.34) DISPLACEMENT FOR SIMPLY-SUPPORTED THICK (h=2) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER CONCENTRATED LOADING.

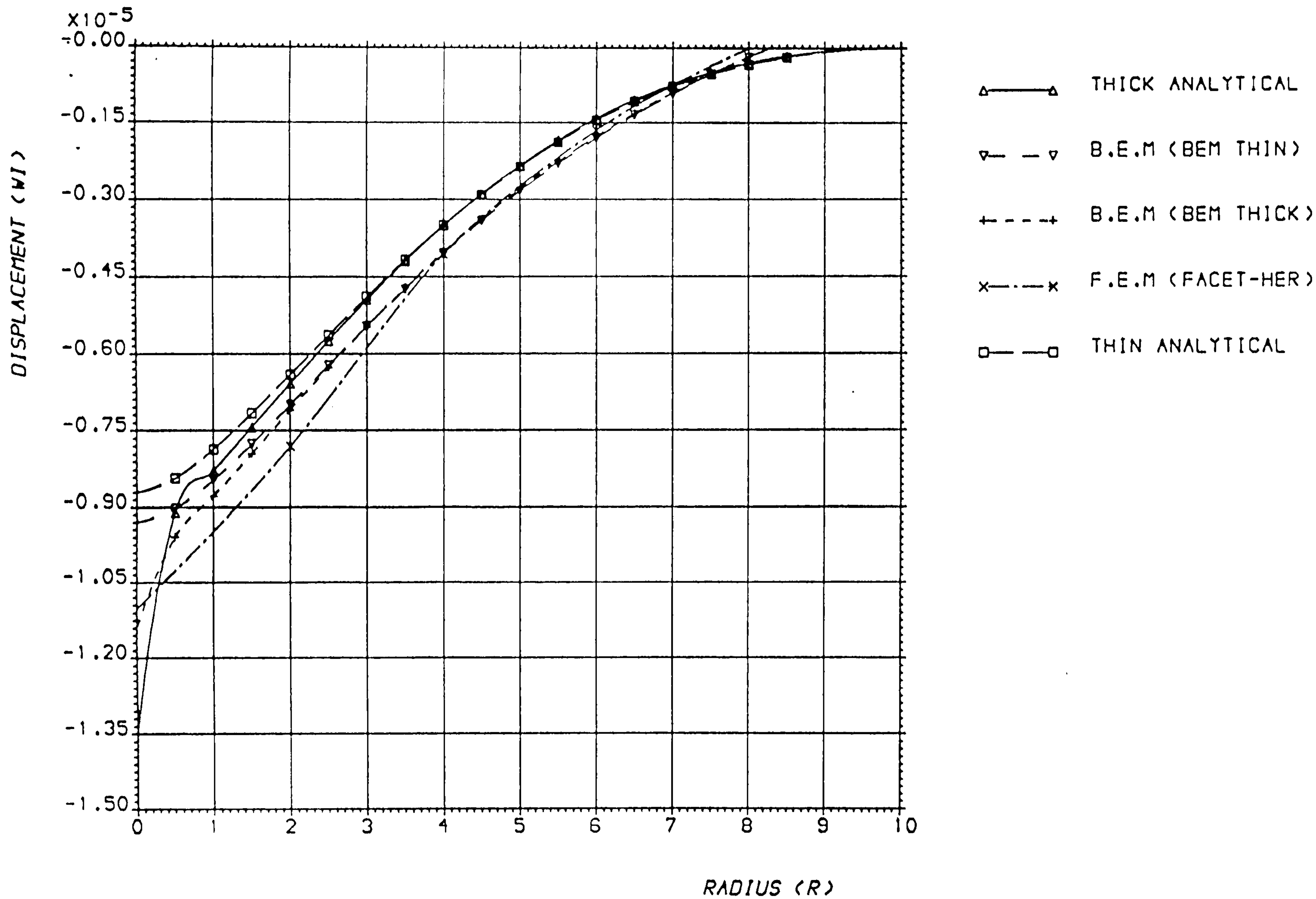


FIG (7.35) DISPLACEMENT DISTRIBUTION FOR FREE THIN (h=1) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER CONCENTRATED LOADING.

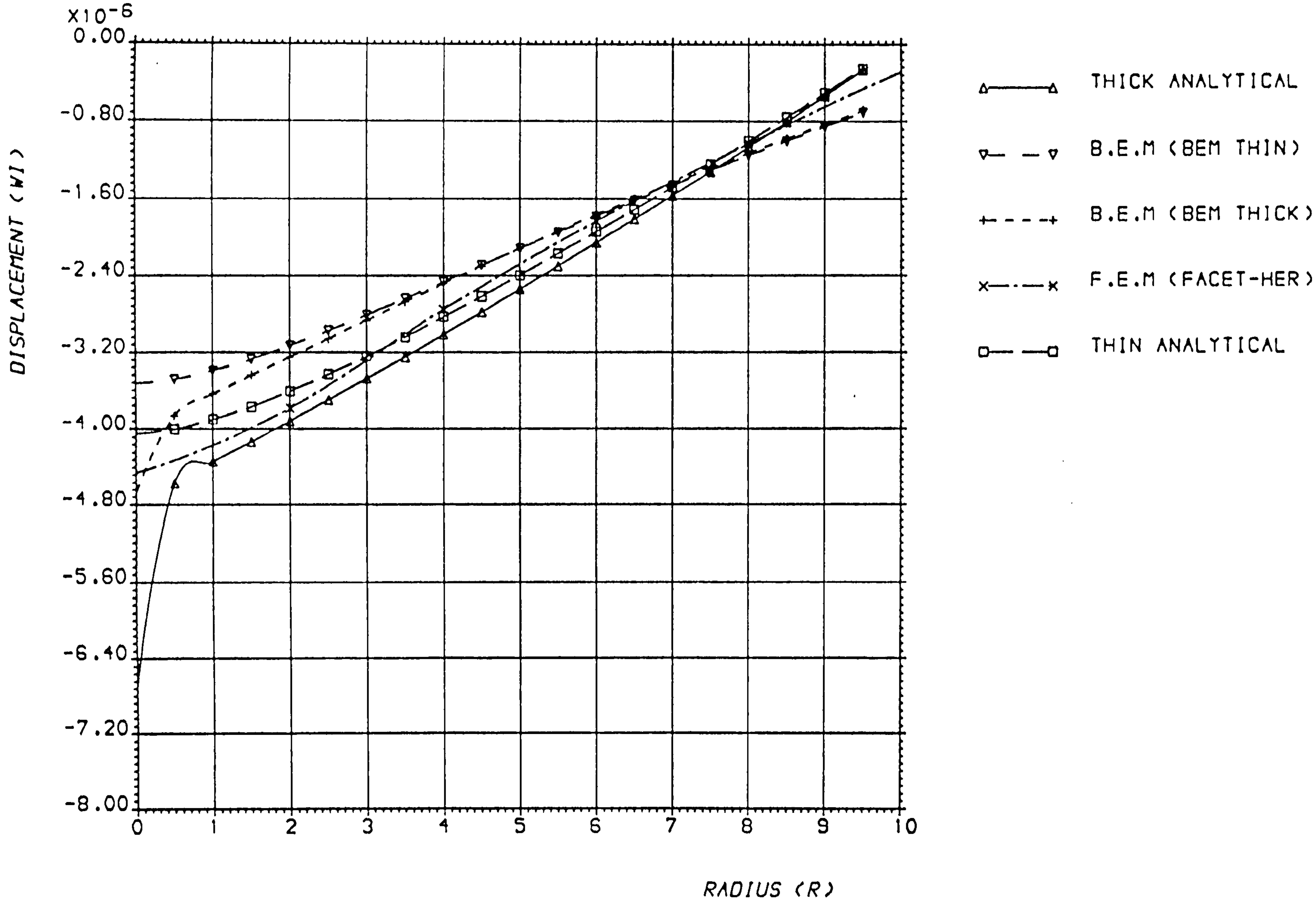


FIG (7.36) DISPLACEMENT DISTRIBUTION FOR FREE THICK (h=2) CIRCULAR PLATE ON ELASTIC FOUNDATION UNDER CONCENTRATED LOADING.

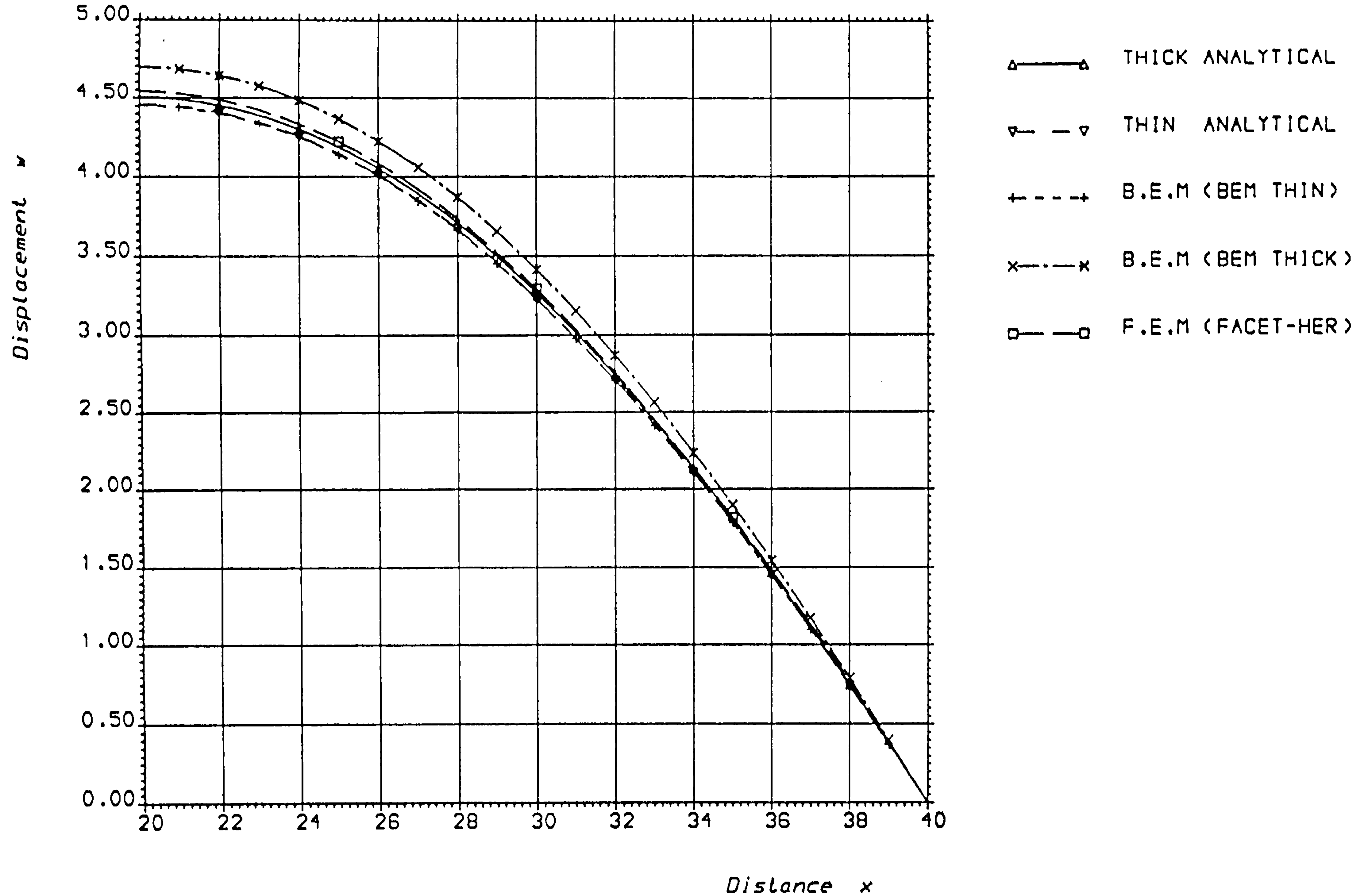


FIG (7.37) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE (DOMAIN LOADING CASE WITH FOUNDATION K=200, THICKNESS h=2)

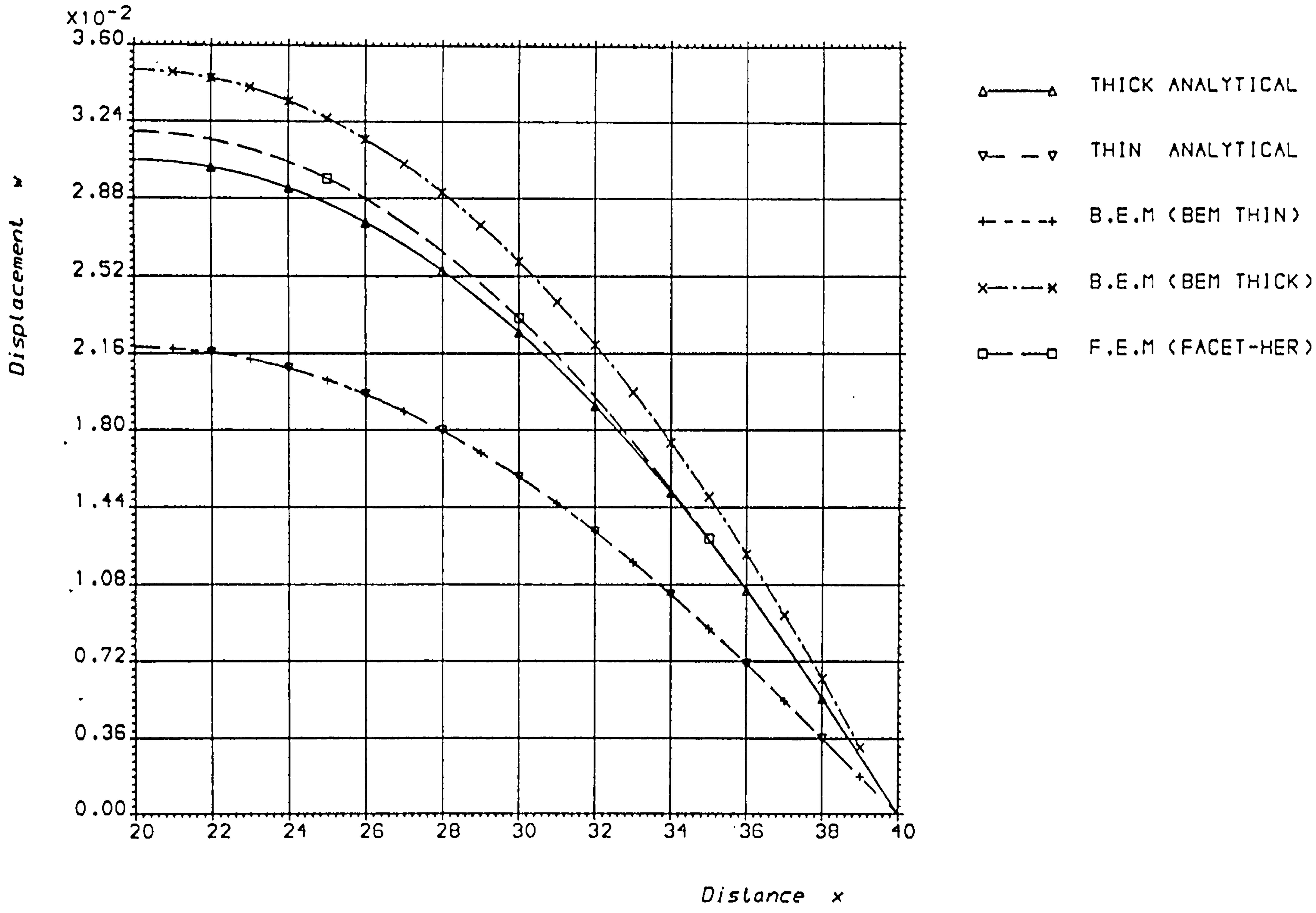


FIG (7.38) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE (DOMAIN LOADING CASE WITH FOUNDATION K=200, THICKNESS h=12)

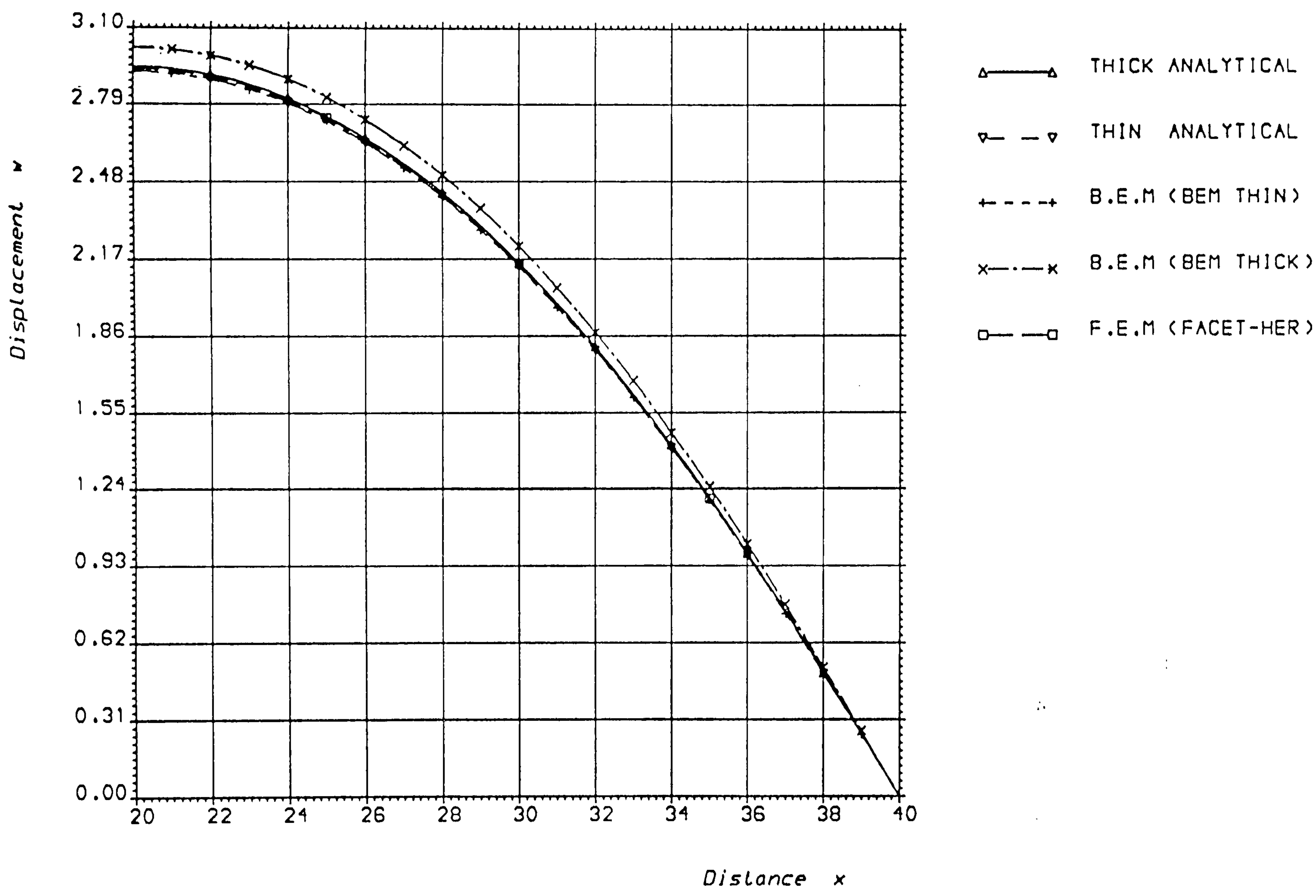


FIG (7.39) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP. RECTANGULR PLATE
(DOMAIN LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=2)

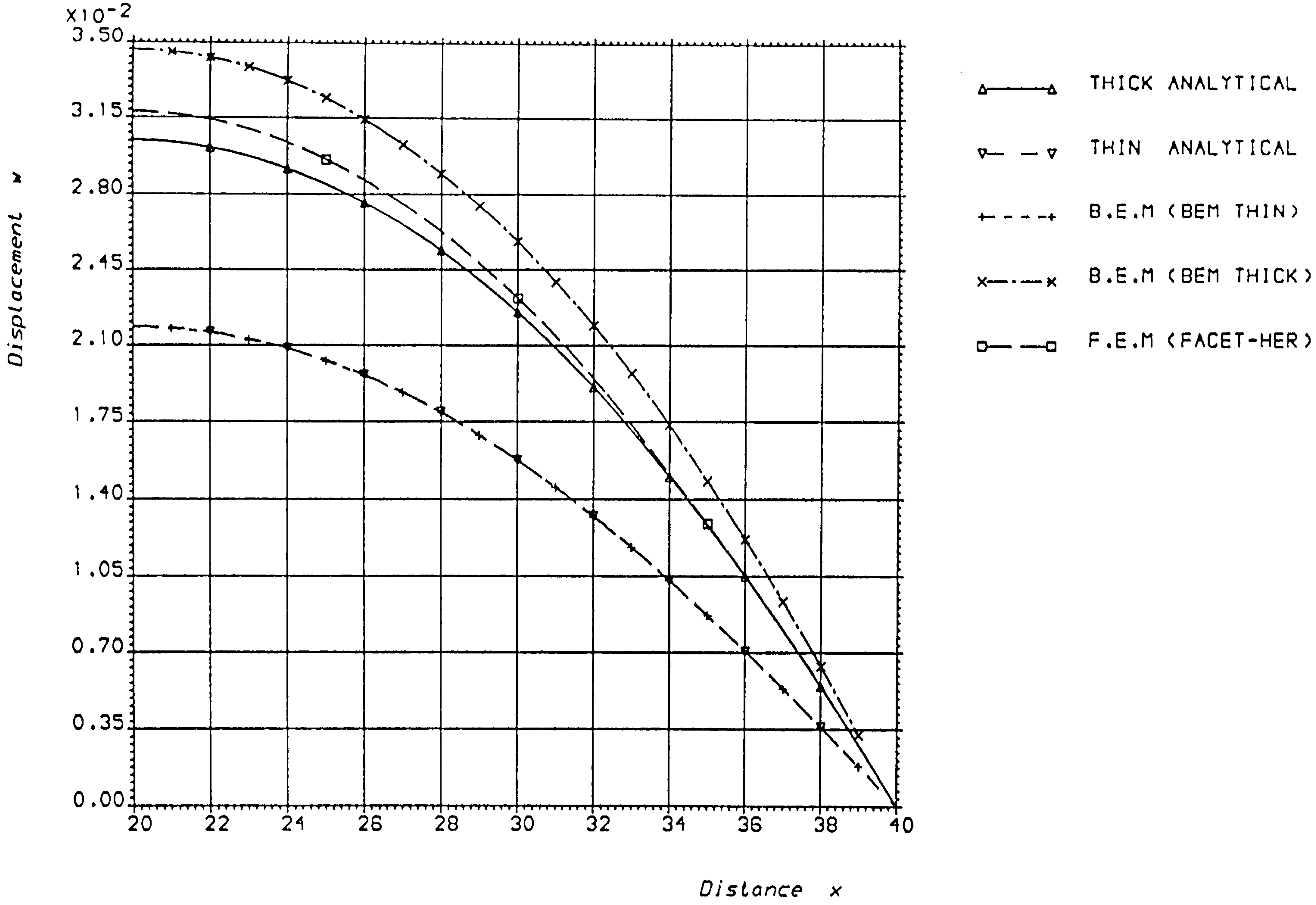


FIG (7.40) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUPP. RECTANGULR PLATE
(DOMAIN LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=12)

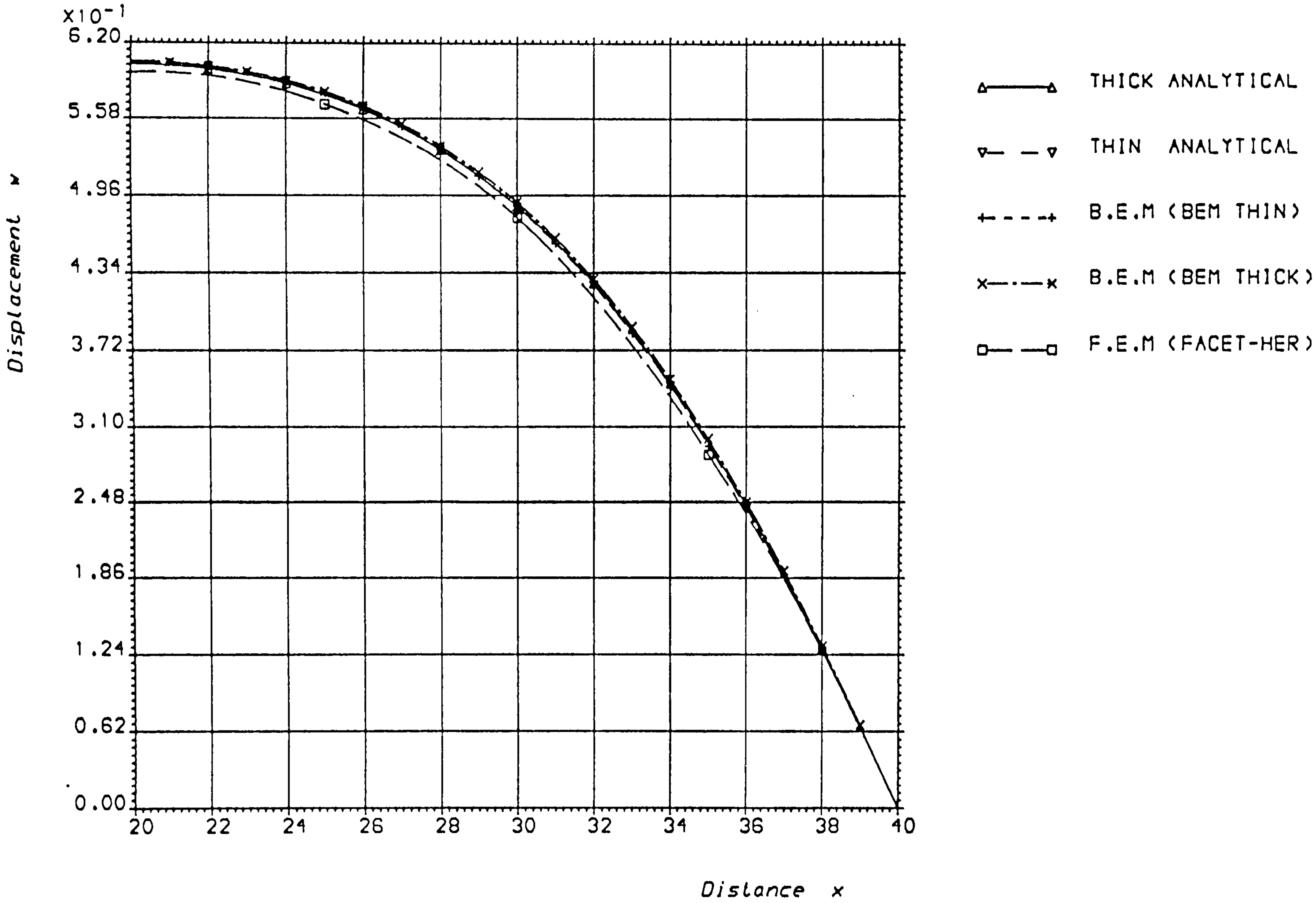


FIG (7.41) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP. RECTANGULR PLATE (DOMAIN LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=2)

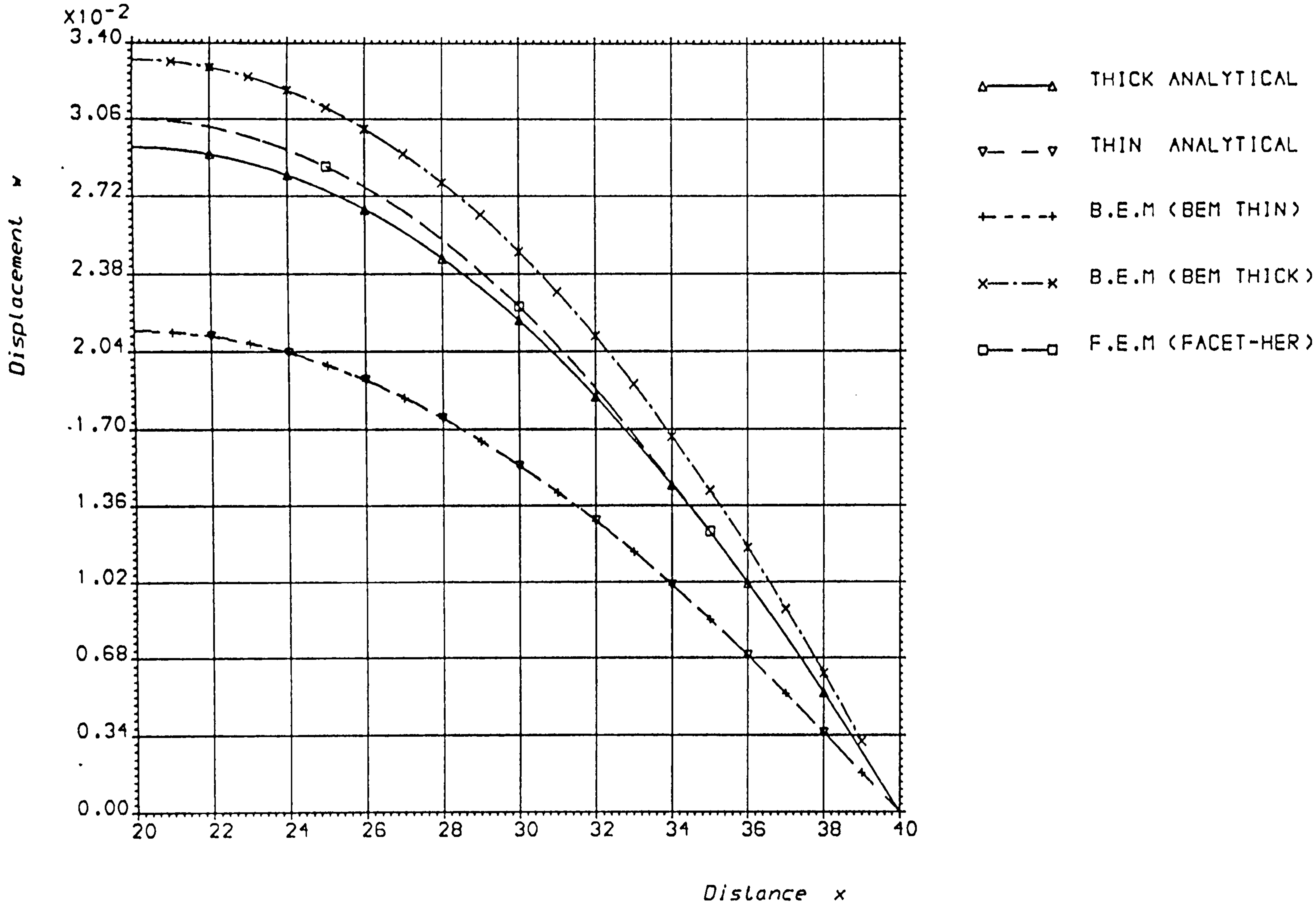


FIG (7.42) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUPP. RECTANGULR PLATE (DOMAIN LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=12)

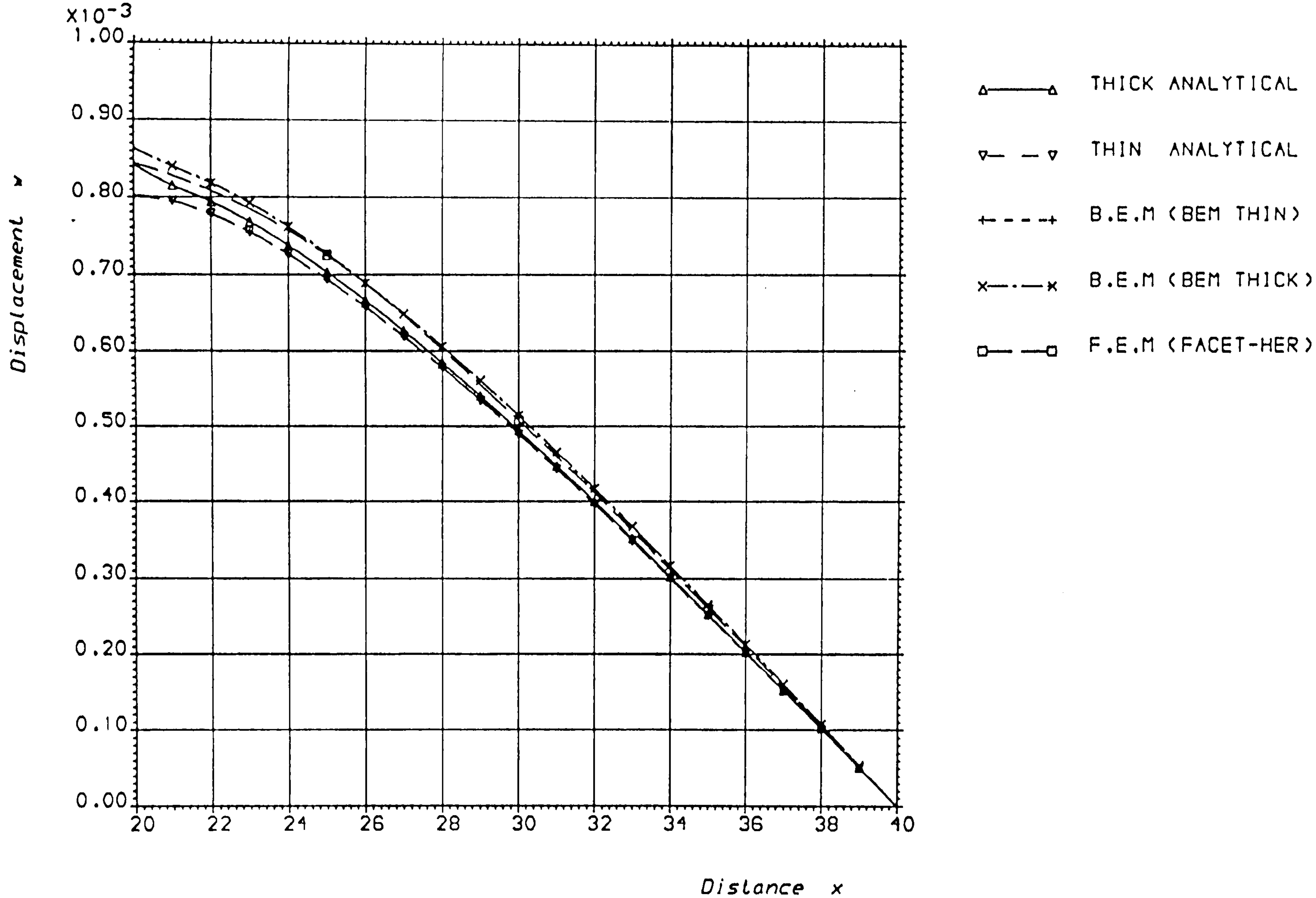


FIG (7.43) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE (CONC. LOADING CASE WITH FOUNDATION K=200, THICKNESS h=2)

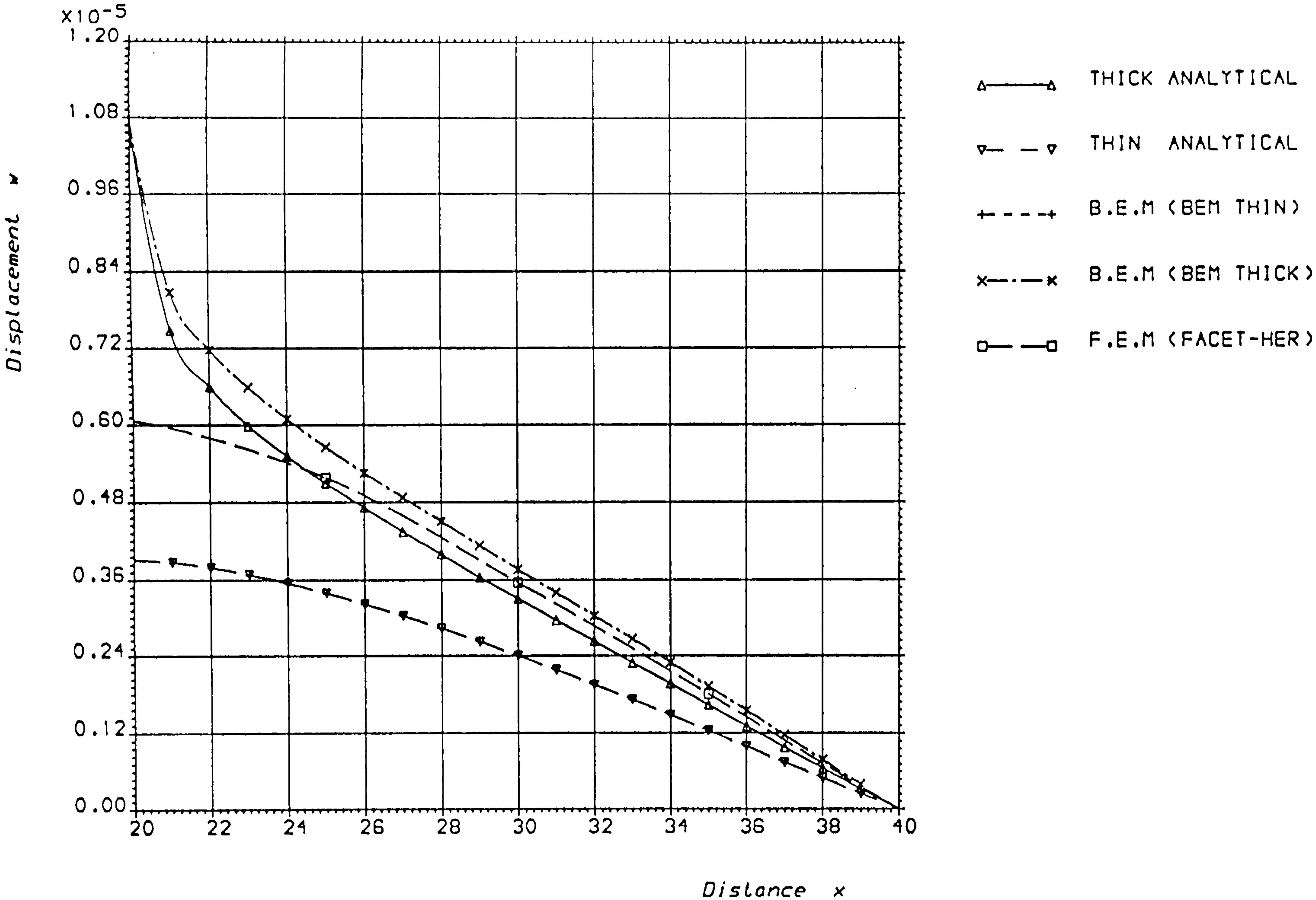


FIG (7.44) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(CONC. LOADING CASE WITH FOUNDATION K=200, THICKNESS h=12)

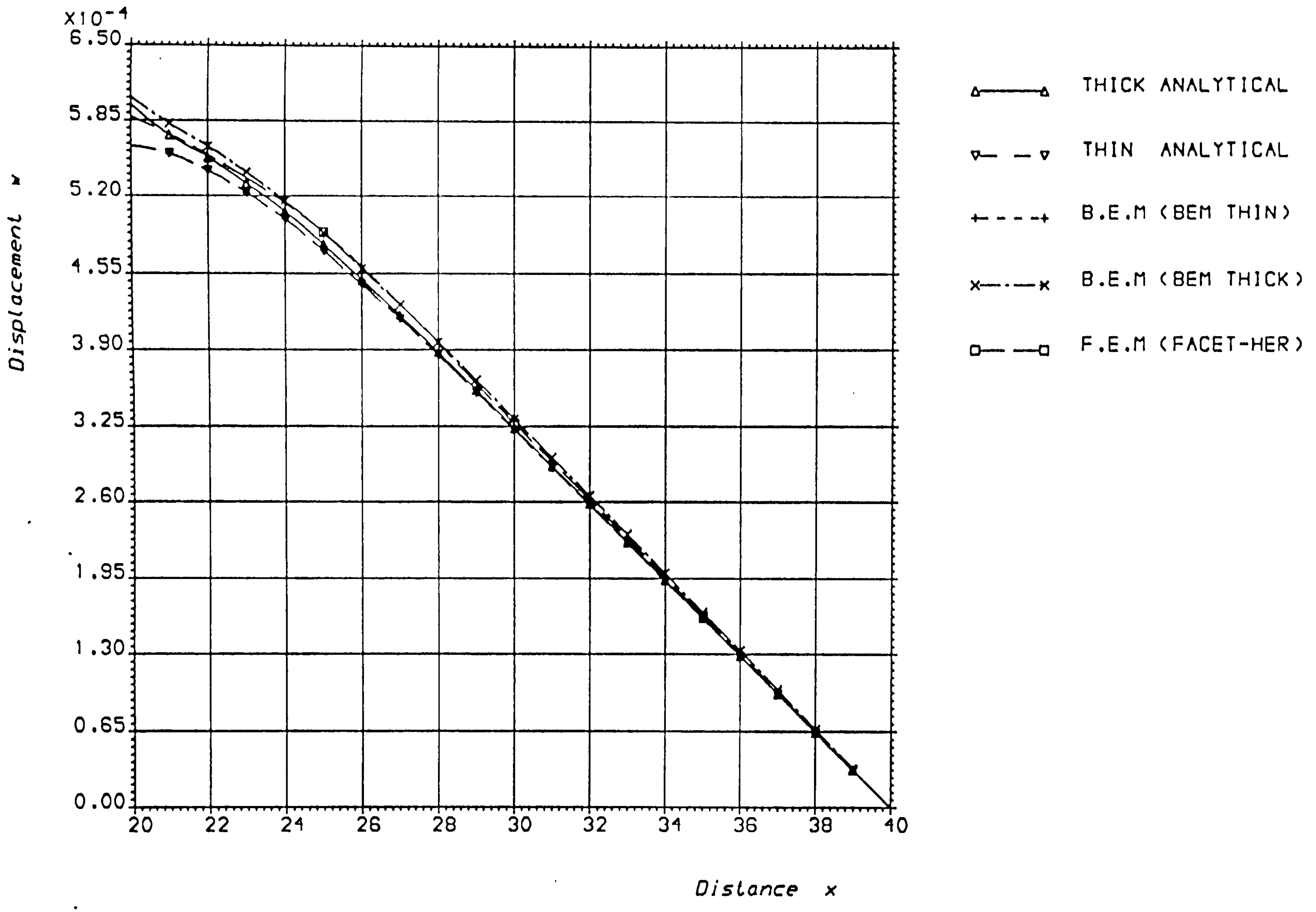


FIG (7.45) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(CONC. LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=2)

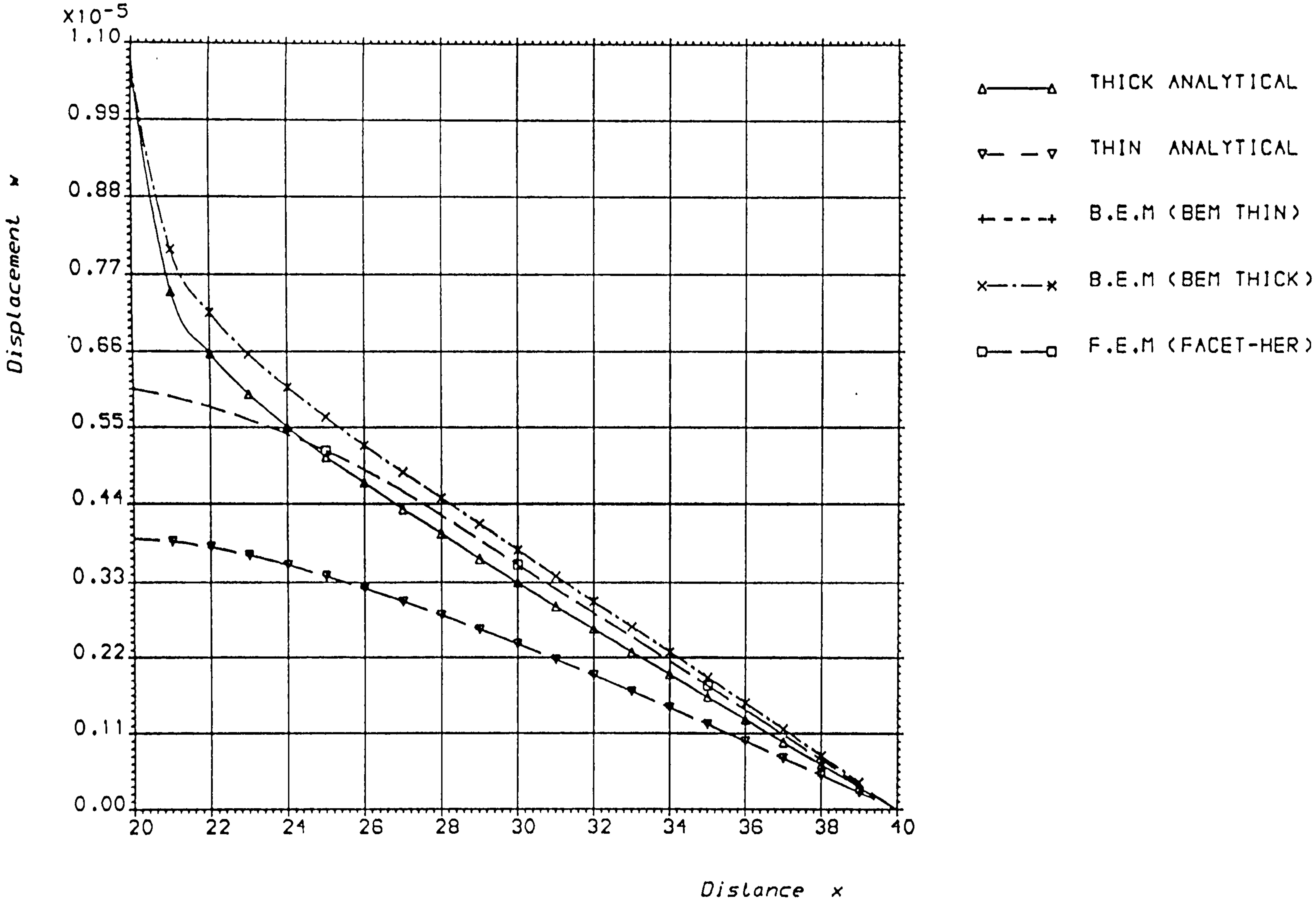


FIG (7.46) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE (CONC. LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=12)

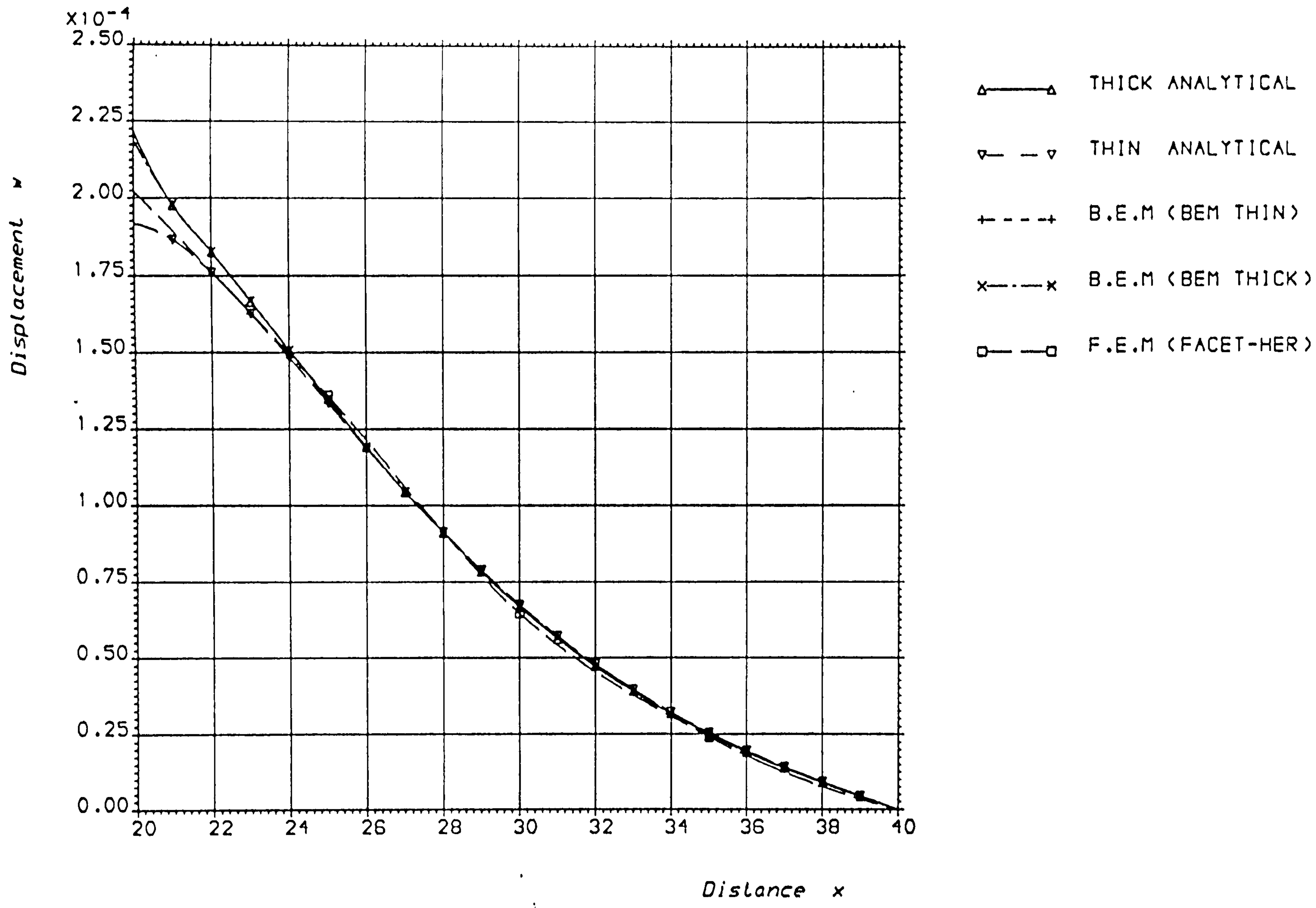


FIG (7.47) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE (CONC. LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=2)

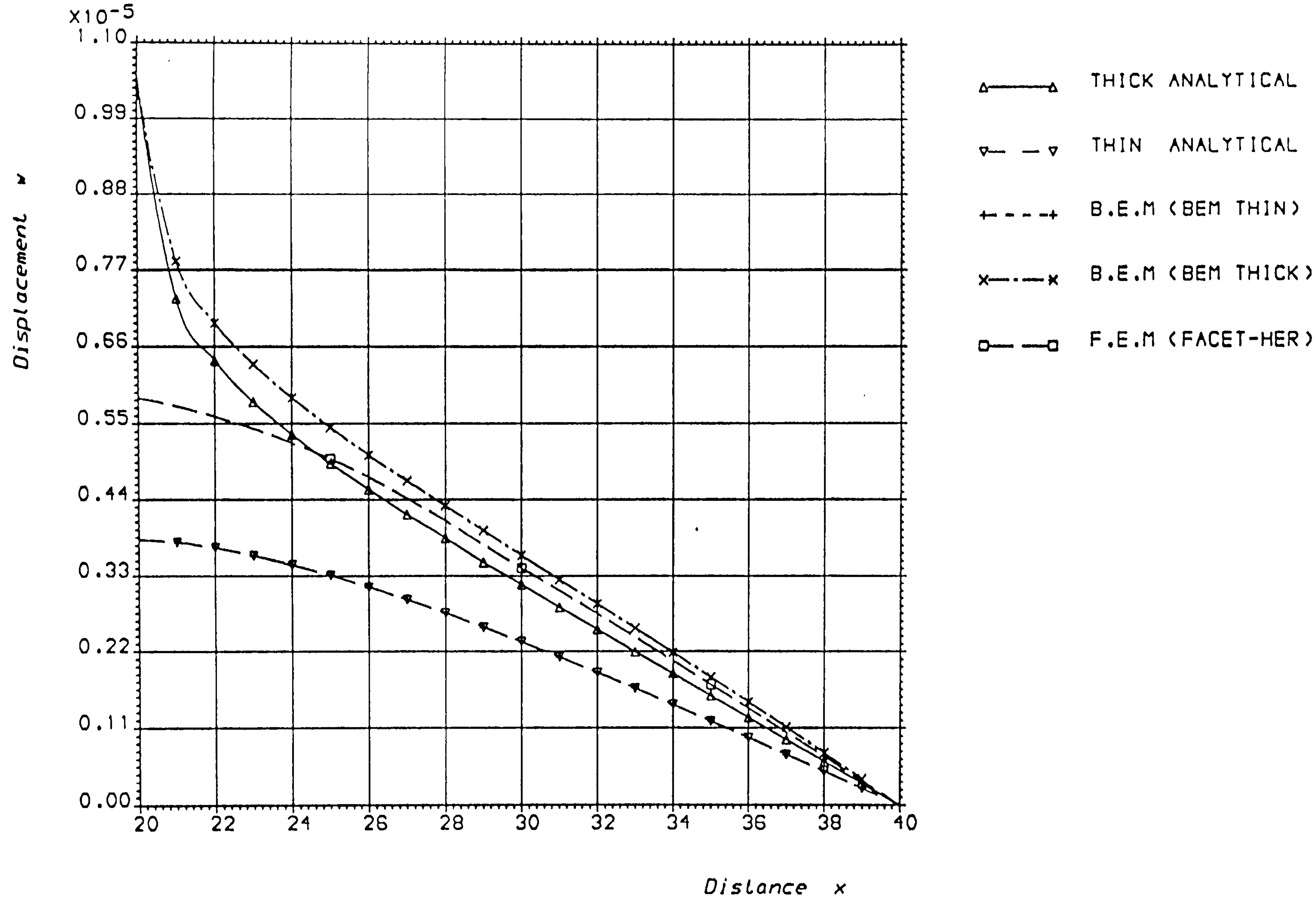


FIG (7.48) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE (CONC. LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=12)

CHAPTER EIGHT

CONCLUSIONS

8.1 CONCLUSIONS

It is clear from the previous chapters that a number of original derivations have been developed and by using a comprehensive set of case studies such derivations have proved to be accurate and reliable for the analysis of thin and thick plates on elastic foundations.

The main points of originality derived and validated in this work can be summarised as follows:

(i) The derivation of high-order shear finite element for thin and thick plates on elastic foundations.

(ii) Different derivations of 3 degrees-of-freedom boundary element analysis of thin plates on elastic foundations including the use of "Modified" Kelvin functions as defined in this work.

(iii) The full derivation of boundary element theory for thick plates on elastic foundations including the derivation of several possible fundamental solutions using Fourier and Hankel integral transforms.

(iv) The implementation of fictitious boundary concept which leads to the elimination of singular and divergent integral terms in the boundary integral equations.

(v) The derivation of some analytical solutions for thin and thick circular and rectangular plates on elastic foundations.

From the case studies analysed, the following remarks can be concluded.

(i) Generally speaking, the boundary element method has proved to be more accurate than the finite element method for the analysis of thin and thick plates on elastic foundations.

(ii) The boundary element analysis of plates with free edge conditions is only possible by means of programs which use fictitious boundaries.

(iii) The boundary element program based on domain integrals has given poor results for the analysis of thick plates, compared to the one which employs direct boundary integrals.

(iv) Finite elements with mid-side nodes should be used with Mindlin first order program in order to obtain acceptable results.

8.2 RECOMMENDATIONS FOR FUTURE WORK

(i) An early objective of the current author was to analyse curved shells with and without elastic foundations. Some work on the finite element programs has already been carried out but it is interesting to extend the boundary element analysis so as to deal with curved shells and folded plates.

(ii) The derivation of fundamental solutions by means of integral transforms could be extended to orthotropic plates and it would then become possible to derive boundary integral equations for thin and thick plates made of orthotropic materials.

APPENDIX A

Appendix A

LAGRANGIAN SHAPE FUNCTION FOR QUADRILATERAL AND TRIANGULAR ELEMENTS

Using the Lagrangian bivariate interpolation formula [Ref 62 ,63], the Lagrangian shape functions can be written as follows:

a) 4–Noded quadrilateral element:

$$N_1=(1-\zeta)(1-\eta)$$

$$N_2=\zeta(1-\eta)$$

$$N_3=\zeta\eta$$

$$N_4=(1-\zeta)\eta$$

b) 8–Noded quadrilateral element:

$$N_1=(1-\zeta)(1-\eta)(1-2\zeta-2\eta)$$

$$N_2=4\zeta(1-\zeta)(1-\eta)$$

$$N_3=-\zeta(1-\eta)(1-2\zeta+2\eta)$$

$$N_4=4\zeta\eta(1-\eta)$$

$$N_5=-\zeta\eta(3-2\zeta-2\eta)$$

$$N_6=4\zeta\eta(1-\zeta)$$

$$N_7=-\eta(1-\zeta)(1-2\zeta-2\eta)$$

$$N_8=4\eta(1-\zeta)(1-\eta)$$

c) 9–Noded quadrilateral element:

$$N_1=(1-\zeta)(1-\eta)(1-2\zeta-2\eta)$$

$$N_2=4\zeta(1-\zeta)(1-\eta)$$

$$N_3=-\zeta(1-\eta)(1-2\zeta-2\eta)$$

$$N_4=4\zeta\eta(1-\eta)$$

$$N_5 = -\zeta\eta(3-2\zeta-2\eta)$$

$$N_6 = 4\zeta\eta(1-\zeta)$$

$$N_7 = \eta(1-\zeta)(1-2\zeta+2\eta)$$

$$N_8 = 4\eta(1-\zeta)(1-\eta)$$

$$N_9 = (4\zeta)(4\eta)(1-\zeta)(1-\eta)$$

d) <u>3–Noded triangular element:</u>

$$N_1 = 1-\zeta-\eta$$

$$N_2 = \zeta$$

$$N_3 = \eta$$

c) <u>6–Noded triangular element:</u>

$$N_1 = (1-\zeta-\eta)(1-2\zeta-2\eta)$$

$$N_2 = 4\zeta(1-\zeta-\eta)$$

$$N_3 = -\eta(1-2\zeta)$$

$$N_4 = 4\zeta\eta$$

$$N_5 = -\eta(1-2\eta)$$

$$N_6 = 4\eta(1-\zeta-\eta)$$

APPENDIX B

APPENDIX B

SECOND ORDER INTRINSIC DERIVATIONS FOR QUADRILATERAL AND TRIANGULAR HERMITIAN SHAPE FUNCTIONS

a) Quadrilateral Hermitian shape functions second order intrinsic derivatives.

$$\frac{\partial^2 N_1}{\partial\zeta^2}=(-6+12\zeta)(1-\eta)$$

$$\frac{\partial^2 N_1}{\partial\eta^2}=(-6+12\eta)(1-\zeta)$$

$$\frac{\partial^2 N_1}{\partial\zeta\partial\eta}=6\zeta(1-\zeta)+6\eta(1-\eta)-1$$

$$\frac{\partial^2 N_2}{\partial\zeta^2}=J_{12}(-4+6\zeta)(1-\eta)$$

$$\frac{\partial^2 N_2}{\partial\eta^2}=J_{22}(-4+6\eta)(1-\zeta)$$

$$\frac{\partial^2 N_2}{\partial\zeta\partial\eta}=-J_{12}(1-4\zeta+3\zeta^2)-J_{22}(1-4\eta+3\eta^2)$$

$$\frac{\partial^2 N_3}{\partial\zeta^2}=J_{11}(6\zeta-4)(1-\eta)$$

$$\frac{\partial^2 N_3}{\partial\eta^2}=-J_{21}(6\eta-4)(1-\zeta)$$

$$\frac{\partial^2 N_3}{\partial\zeta\partial\eta}=J_{11}(1-4\zeta+3\zeta^2)+J_{21}(1-4\eta+3\eta^2)$$

$$\frac{\partial^2 N_4}{\partial\zeta^2}=(6-12\zeta)(1-\eta)$$

$$\frac{\partial^2 N_4}{\partial\eta^2}=(6-12\eta)(1-\zeta)$$

$$\frac{\partial^2 N_4}{\partial\zeta\partial\eta}=6\zeta(\zeta-1)+6\eta(\zeta-1)+1$$

$$\frac{\partial^2 N_5}{\partial\zeta^2}=J_{12}(6\zeta-2)(1-\eta)$$

$$\frac{\partial^2 N_5}{\partial\eta^2}=J_{22}(6\eta-4)\zeta$$

$$\frac{\partial^2 N_5}{\partial\zeta\partial\eta}=J_{12}(2\zeta-3\zeta^2)+J_{22}(1-4\eta+3\eta^2)$$

$$\frac{\partial^2 N_6}{\partial \zeta^2}=J_{11}(2-6\zeta)(1-\eta)$$

$$\frac{\partial^2 N_6}{\partial \eta^2}=-J_{21}(6\eta-4)$$

$$\frac{\partial^2 N_6}{\partial \zeta \partial \eta}=J_{11}(-2\zeta+3\zeta^2)-J_{21}(1-4\eta+2\eta^2)$$

$$\frac{\partial^2 N_7}{\partial \zeta^2}=(6-12\zeta)\eta$$

$$\frac{\partial^2 N_7}{\partial \eta^2}=(6-12\eta)\zeta$$

$$\frac{\partial^2 N_7}{\partial \zeta \partial \eta}=6\zeta(1-\zeta)+6\eta(1-\eta)-1$$

$$\frac{\partial^2 N_8}{\partial \zeta^2}=J_{12}(-2+6\zeta)\eta$$

$$\frac{\partial^2 N_8}{\partial \eta^2}=J_{22}(-2+6\eta)\zeta$$

$$\frac{\partial^2 N_8}{\partial \zeta \partial \eta}=J_{12}(-2\zeta+3\zeta^2)+J_{22}(-2\eta+3\eta^2)$$

$$\frac{\partial^2 N_9}{\partial \zeta^2}=J_{11}(-2+6\zeta)\eta$$

$$\frac{\partial^2 N_9}{\partial \eta^2}=J_{21}(-2+6\eta)\zeta$$

$$\frac{\partial^2 N_9}{\partial \zeta \partial \eta}=-J_{11}(-2\zeta+3\zeta^2)-J_{21}(-2\eta+3\eta^2)$$

$$\frac{\partial^2 N_{10}}{\partial \zeta^2}=(-6+12\zeta)\eta$$

$$\frac{\partial^2 N_{10}}{\partial \eta^2}=(6-12\eta)(1-\zeta)$$

$$\frac{\partial^2 N_{10}}{\partial \zeta \partial \eta}=6\zeta(\zeta-1)+6\eta(\zeta-1)+1$$

$$\frac{\partial^2 N_{11}}{\partial \zeta^2}=J_{12}\zeta(-4+6\zeta)$$

$$\frac{\partial^2 N_{11}}{\partial \eta^2}=J_{22}(-2+6\eta)(1-\zeta)$$

$$\frac{\partial^2 N_{11}}{\partial \zeta \partial \eta}=J_{12}(1-4\zeta+3\zeta^2)-J_{22}(-2\eta+3\eta^2)$$

$$\frac{\partial^2 N_{12}}{\partial \zeta^2} = -J_{11}(-9+6\zeta)\eta$$

$$\frac{\partial^2 N_{12}}{\partial \eta^2} = -J_{21}(-2+6\eta)(1-\zeta)$$

$$\frac{\partial^2 N_{12}}{\partial \zeta \partial \eta} = -J_{11}(1-4\zeta+3\zeta^2)+J_{21}(-2\eta+3\eta^2)$$

b) Triangular element Hermitian shape functions second order intrinsic derivatives

$$\frac{\partial^2 N_1}{\partial \zeta^2} = -6+12\zeta+8\eta$$

$$\frac{\partial^2 N_1}{\partial \eta^2} = -6+12\eta+8\zeta$$

$$\frac{\partial^2 N_1}{\partial \zeta \partial \eta} = -4+8\zeta+8\eta$$

$$\frac{\partial^2 N_2}{\partial \zeta^2} = J_{12}(-4+6\zeta+3\eta)+J_{12}\eta$$

$$\frac{\partial^2 N_2}{\partial \eta^2} = J_{22}(-4+6\eta+3\zeta)+J_{12}\zeta$$

$$\frac{\partial^2 N_2}{\partial \zeta \partial \eta} = J_{12}(-1.5+3\zeta+\eta)+J_{22}(-1.5+3\eta+\zeta)$$

$$\frac{\partial^2 N_3}{\partial \zeta^2} = -J_{11}(6\zeta-4+3\eta)-J_{21}\eta$$

$$\frac{\partial^2 N_3}{\partial \eta^2} = -J_{21}(6\eta-4+3\zeta)-J_{11}\zeta$$

$$\frac{\partial^2 N_3}{\partial \zeta \partial \eta} = -J_{11}(-1.5+3\zeta+\eta)-J_{21}(-1.5+3\eta+\zeta)$$

$$\frac{\partial^2 N_4}{\partial \zeta^2} = 6-12\zeta-4\eta$$

$$\frac{\partial^2 N_4}{\partial \eta^2} = -4\zeta$$

$$\frac{\partial^2 N_4}{\partial \zeta \partial \eta} = 2-4\zeta-4\eta$$

$$\frac{\partial^2 N_5}{\partial \zeta^2} = -J_{12}(2-6\zeta-3\eta)+(J_{22}-J_{12})\eta$$

$$\frac{\partial^2 N_5}{\partial \eta^2}=J_{12}\zeta-(J_{22}-J_{12})\zeta$$

$$\frac{\partial^2 N_5}{\partial \zeta \partial \eta}=J_{12}(0.5-3\zeta-\eta)-(J_{22}-J_{12})(0.5+\zeta-\eta)$$

$$\frac{\partial^2 N_6}{\partial \zeta^2}=J_{11}(2-6\zeta-3\eta)+(J_{22}-J_{12})\eta$$

$$\frac{\partial^2 N_6}{\partial \eta^2}=-J_{11}\zeta+(J_{21}-J_{11})\zeta$$

$$\frac{\partial^2 N_6}{\partial \zeta \partial \eta}=J_{11}(0.5-3\zeta-\eta)-(J_{21}-J_{11})(0.5+\zeta-\eta)$$

$$\frac{\partial^2 N_7}{\partial \zeta^2}=-4\eta$$

$$\frac{\partial^2 N_7}{\partial \eta^2}=6-12\eta-4\zeta$$

$$\frac{\partial^2 N_7}{\partial \zeta \partial \eta}=2-4\zeta-4\eta$$

$$\frac{\partial^2 N_8}{\partial \zeta^2}=J_{22}\zeta-(J_{12}-J_{22})\eta$$

$$\frac{\partial^2 N_8}{\partial \eta^2}=-J_{22}(2-3\zeta-6\eta)+(J_{12}-J_{22})\zeta$$

$$\frac{\partial^2 N_8}{\partial \zeta \partial \eta}=-J_{22}(0.5-\zeta-3\eta)+(J_{12}-J_{22})(0.5-\zeta+\eta)$$

$$\frac{\partial^2 N_9}{\partial \zeta^2}=-J_{21}\eta+(J_{11}-J_{21})\eta$$

$$\frac{\partial^2 N_9}{\partial \eta^2}=J_{21}(2-3\zeta-6\eta)-(J_{11}-J_{21})\zeta$$

$$\frac{\partial^2 N_9}{\partial \zeta \partial \eta}=J_{21}(0.5-\zeta-3\eta)-(J_{11}-J_{21})(0.5-\zeta-\eta)$$

APPENDIX C

APPENDIX C

PROPERTIES OF BESSEL FUNCTIONS

1–Bessel functions of first and second kinds

Consider the general solution of the following differential equation.

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+(\kappa^2-\frac{\nu^2}{x^2})y=0 \tag{1}$$

it can have one of the following solutions:

(i) For ν = n = an integer value:

$$y(x) = AJ_n(z) + BY_n(z)$$

(ii) For ν having non–integer values:

$$y(x) = AJ_\nu(z) + BY_\nu(z)$$

or

$$y(x) = AJ_\nu(z) + BJ_{-\nu}(z)$$

where

A, B are integration constant,

$Z=\kappa x$

J_ν Bessel function of 1^{st} kind and ν^{th} order.
Y_ν Bessel function of 2^{nd} kind and ν^{th} order.

1.1 Definition of Bessel functions in terms of infinite series

$$J_{\pm\nu}(z) = \sum_{r=0}^{\infty}(-1)^r\frac{(z/2)^{2r\pm\nu}}{r!\Gamma(r+1\pm\nu)} \tag{2}$$

$$\underset{n}{J}(z) = \sum_{r=0}^{\infty}(-1)^r\frac{(z/2)^{2r+n}}{r!(r+n)!} \tag{3}$$

$$J_{-n} = (-1)^n J_n(z) \tag{4}$$

$$Y_n(z) = \frac{2}{\pi}\log\left(\frac{z}{2}\right) J_n(z) - \frac{(z/2)^{-n}}{\pi} \sum_{s=0}^{n-1}\frac{(n-s-1)!}{s!}\left(\frac{z}{2}\right)^{2s}$$

$$- \frac{(z/2)^n}{\pi} \sum_{k=0}^{\infty}\left\{\psi(k+1)+\psi(n+k1)\right\}\frac{\left(-\frac{z^2}{4}\right)^k}{k!(n+k)!} \tag{5}$$

where

$$\psi(1)=-\gamma \tag{6a}$$

$$\psi(n)=-\gamma+\sum_{k=1}^{n-1}\left(\frac{1}{k}\right) \tag{6b}$$

$$\gamma=\lim_{m\to 0}\left\{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}-\log m\right\} = 0.5772\ 15664\ 90153\ 28606\ \cdots \tag{7}$$

Special cases

$$J_o(z) = 1 - \frac{\frac{z^2}{4}}{(1!)^2} + \frac{\left(\frac{z^2}{4}\right)^2}{(2!)^2} - \frac{\left(\frac{z^2}{4}\right)^3}{(3!)^2} + \cdots$$

$$Y_o(z) = \frac{2}{\pi}\left\{\log\left(\frac{z}{2}\right) + \gamma\right\}J_o(z) + \frac{2}{\pi}\left\{\frac{\frac{z^2}{4}}{(1!)^2} - \left(1+\frac{1}{2}\right)\frac{\left(\frac{z^2}{4}\right)^2}{(2!)^2} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{\left(\frac{z^2}{4}\right)^3}{(3!)^2}+\cdots\right\}$$

1.2 Bessel functions of the 3^{rd} kind Hankel functions

$$H_\nu^{(1)}(z)=J_\nu(z)+iY_\nu(z) \tag{8}$$

$$H_\nu^{(2)}(z)=J_\nu(z)-iY_\nu(z) \tag{9}$$

an independent solution of (1) is:

$$y=A\ H_\nu^{(1)}(z)+B\ H_\nu^{(2)}(z) \tag{10}$$

1.3 Recurrence relations:

$$B_{\nu-1}(z) + B_{\nu+1}(z) = \frac{2\nu}{z}B_\nu(z)$$

$$B_{\nu-1}(z) - B_{\nu+1}(z) = 2B'_\nu(z)$$

$$B'_\nu(z) = B_{\nu-1}(z) - \frac{\nu}{z}B_\nu(z)$$

$$B'_\nu(z) = -B_{\nu+1}(z) + \frac{\nu}{z}B_\nu(z)$$

where

B denotes J, Y, $H^{(1)}$ and $H^{(2)}$ or any linear combination of those functions.

$$J'_o(z) = -J_1(z)$$

$$Y'_o(z) = -Y_1(z)$$

1.4 Higher order derivatives

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\left\{z^k B_\nu(z)\right\}=z^{\nu-k}B_{\nu-k}(z) \tag{11a}$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\left\{z^{-\nu}B_\nu(z)\right\}=(-1)^k z^{-\nu-k}B_{\nu+k}(z) \tag{11b}$$

$$B_\nu^{(k)}(z) = \frac{1}{z^k}\left\{B_{\nu-k}(z) - \binom{K}{1}B_{\nu-k+2}(z)+ \binom{K}{1}B_{\nu-k+4}(z)+\cdots+(-1)^k B_{\nu+k}(z)\right\}$$

$$k=0,\ 1,\ 2,\ \cdots$$

2- Modified Bessel functions

The solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-(k^2+\frac{\nu^2}{x^2})y=0 \tag{12}$$

is

$$y=A\ I_\nu(z)+B\ K_\nu(z)$$

$$= A\ I_\nu(z)+B\ I_\nu(z)$$

always ν is non−integer

$I_\nu(z)$ is the modified Bessel function of the 1^{st} kind

$K_\nu(z)$ is the modified Bessel function of the 2^{nd} kind

and

$$z=kx$$

2.1 Infinite series

$$I_n(z) = i^{-n}J_n(iz) = \sum_{k=0}^{\infty}\frac{(z/2)^{n+2k}}{k!(n+k)!} \tag{13}$$

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty}\frac{(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)} \tag{14}$$

$$K_n(z) = \frac{1}{2}\left(\frac{z}{2}\right)^{-n} \sum_{s=0}^{n-1}\frac{(n-s-1)!}{s!}\left(\frac{-z^2}{4}\right)^s + (-1)^{n+1}\log\left(\frac{z}{2}\right)I_n(z)$$

$$+\frac{(-1)^n}{2}\left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty}\left\{\left(\psi(k+1) + \psi(n+k+1)\right)\frac{(z/2)^{2k}}{k!(n+k)!} + \cdots\right\} \tag{15}$$

Special cases:

$$I_o(z)=1+\frac{z^2/4}{(1!)^2}+\frac{(z^2/4)^2}{(2!)^2}+\frac{(z^2/4)^3}{(3!)^2}+\cdots$$

$$I_1(z)=\frac{z}{2}\left\{1+\frac{z^2/4}{2(1!)^2}+\frac{(z^2/4)^2}{3(2!)^2}+\frac{(z^2/4)^3}{4(3!)^2}+\cdots\right\}$$

$$K_o(z)=-[\log\left(\frac{z}{2}\right)+\gamma]I_o(z)+\frac{z^2/4}{2(1!)^2}+(1+\frac{1}{2})\frac{(z^2/4)^2}{(2!)^2}+(1+\frac{1}{2}+\frac{1}{3})\frac{(z^2/4)^3}{(3!)^2}+\cdots$$

$$K_1(z)=[\log\left(\frac{z}{2}\right)+\gamma]I_1(z)+\frac{1}{2}\left(\frac{z}{2}\right)^{-1}-\frac{1}{2}\sum_{k=0}^{\infty}\frac{(z/2)^{2k+1}}{k!(k+1)!}\left(\phi(k)+\phi(k+1)\right)$$

where

$$\phi(0)=0$$

$$\phi(\rho)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\rho}$$

Notice that

$$-k^2=i^2k^2$$

and for n being an integer:

$$I_n(z)=i^{-n}J_n(iz)$$

2.2 Recurrence relations

$$M_{\nu-1}(z) - M_{\nu+1}(z) = \frac{2\nu}{z}M_\nu(z)$$

$$M_{\nu-1}(z) + M_{\nu+1}(z) = 2M'_\nu(z)$$

$$M'_\nu(z) = M_{\nu-1}(z) - \frac{\nu}{z}M_\nu(z)$$

$$M'_\nu(z) = M_{\nu+1}(z) + \frac{\nu}{z}M_\nu(z)$$

where M_ν represents I_ν, $e^{\nu\pi i}K_\nu$ or any linear combination of any of them.

For $K_n(z)$:

$$K_{n+1}(z)=K_{n-1}(z)+\frac{2n}{z}K_n(z) \tag{17a}$$

$$K'_n(z)=-\frac{1}{2}\left\{K_{n-1}(z)+K_{n+1}(z)\right\}$$

$$=-\left\{K_{n-1}(z)+\frac{n}{z}K_n(z)\right\} \tag{17b}$$

Special cases

$$I_o'(z)=I_1(z) \tag{18a}$$

$$K_o'(z)=-K_1(z) \tag{18b}$$

$$K_o''(z)=K_o(z)+\frac{1}{z}K_1(z) \tag{18c}$$

$$K_1'(z)=-\Big\{K_o(z)+\frac{1}{z}K_1(z)] \tag{18d}$$

2.3 Higher order derivatives

$$\Big(\frac{1}{z}\frac{d}{dz}\Big)^k\Big\{z^{\nu}M_{\nu}(z)\Big\}=z^{\nu-k}M_{\nu+k}(z) \tag{19a}$$

$$\Big(\frac{1}{z}\frac{d}{dz}\Big)^k\Big\{z^{-\nu}M_{\nu}(z)\Big\}=z^{-\nu-k}M_{\nu+k}(z) \tag{19b}$$

$$M_{\nu}^{(k)}(z)=\frac{1}{z^k}\Big\{M_{\nu-k}(z)+\binom{k}{1}M_{\nu-k+2}(z)+\binom{k}{2}M_{\nu-k+4}(z)+\cdots M_{\nu+k}(z)\Big\} \tag{19c}$$

M_{ν} represents $I_{\nu}(z)$ or

$$e^{\nu\pi i}K_{\nu}(z)$$

For integer n

$$\begin{aligned} e^{n\pi i}K_n(z) &=\cos(n\pi)K_n(z) \\ &=(-1)^n K_n(z) \end{aligned} \tag{20}$$

3–Kelvin functions

Defining $\nabla^2=\frac{1}{x}\frac{d}{dx}(x\frac{d}{dx})=\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}$ (21)

equation (1) and (12) may be re–written as follows:

$$\nabla^2 y+(k^2-\frac{\nu^2}{x^2})y=0 \tag{22}$$

$$\nabla^2 y-(k^2+\frac{\nu^2}{x^2})y=0 \tag{23}$$

which have the following general solutions respectively:

$$y_1 = J_\nu(z) + BY_\nu(z) \tag{24}$$
$$y_2 = CI_\nu(z) + DK_\nu(z) \tag{25}$$

Hence the following differential equation

$$\left\{\nabla^2 - (k^2 + \frac{\nu^2}{x^2})\right\}\left\{\nabla^2 + (k^2 - \frac{\nu^2}{x^2})\right\}y = 0 \tag{26}$$

has the following general solution

$$y = AJ_\nu(z) + BY_\nu(z) + CI_\nu(z) + DK_\nu(z) \tag{27}$$

where

$$z = kx$$

For the special case of $\nu = 0$,

$$y = AJ_o(z) + BY_o(z) + CI_o(z) + DK_o(z) \tag{28}$$

is the general solution to

$$\nabla^2\nabla^2 y - k^4 y = 0 \tag{29}$$

Consider next the following differential equation

$$\nabla^2\nabla^2 y - \lambda^4 y = 0 \tag{30}$$

using

$$\lambda^4 = -k^4$$

i.e

$$k = \sqrt{i}\lambda \tag{31}$$

and

$$z = kx = \sqrt{i}\lambda x \tag{32}$$

then equation (28) can represent the general solution of equation (30). Consider next the following differential equation

$$\nabla^2 y-(i\lambda^2+\frac{\nu^2}{x^2})y=0 \tag{33}$$

from the previous analysis equation (33) has the following solution

$$y=AI_\nu(\sqrt{i}\lambda x)+BK_\nu(\sqrt{i}\lambda x) \tag{34a}$$

or

$$y=AJ_\nu(i\sqrt{i}\lambda x)+BY_\nu(i\sqrt{i}\lambda x) \tag{34a}$$

Notice that

$$i = e^{(\pi/2)i} = e^{\frac{4n+1}{2}i\pi} \tag{35a}$$

$$\sqrt{i}=e^{(\pi/4)i},\ e^{-3\pi i/4} \tag{35b}$$

$$i\sqrt{i}=e^{(3\pi i/4)},\ e^{-\pi i/4} \tag{35c}$$

substituting into (34) and rearranging real and imaginary parts, then the solution of (33) is any linear combinations of the following

$$b_{er_\nu}(z)+ib_{ei_\nu}(z) \tag{36a}$$
$$b_{er_{-\nu}}(z)+ib_{ei_{-\nu}}(z) \tag{36b}$$
$$K_{er_\nu}(z)+iK_{ei_\nu}(z) \tag{36c}$$
$$K_{er_{-\nu}}(z)+iK_{ei_{-\nu}}(z) \tag{36d}$$

where

$$z=\lambda x$$

where b_{er}, b_{ei}, K_{er}, K_{ei} are Kelvin functions defined such that

$$b_{er_\nu}(z)+ib_{ei_\nu}(z)=J_\nu(ze^{3\pi i/4})$$

$$=e^{\nu\pi i}J_\nu(ze^{-\pi i/4})$$

$$=e^{\frac{\nu\pi i}{2}}I_\nu(ze^{\pi i/4})$$

$$=e^{3\nu\pi i/2}I_{\nu}(ze^{-3\pi i/4}) \tag{37}$$

and

$$K_{er_{\nu}}(z)+iK_{ei_{\nu}}(z)=e^{-\frac{\nu\pi i}{2}}K_{\nu}(xe^{\pi i/4}) \tag{38}$$

subscript ν is usually omitted if $\nu=0$. Hence the general solution of equation (30) is:

$$y=A\ b_{er}(z)+Bb_{ei}(z)+CK_{er}(z)+DK_{ei}(z) \tag{39}$$

where

$$z=\lambda x$$

From previous definitions, it can be deduced that:

$$b_{er}(z)=1-\frac{(z^2/4)^2}{(2!)^2}+\frac{(z^2/4)^4}{(4!)^2}+\cdots \tag{40}$$

$$b_{ei}(z)=\frac{z^2}{4}-\frac{(z^2/4)^3}{(3!)^2}+\frac{(z^2/4)^5}{(5!)^2}+\cdots \tag{41}$$

$$K_{er}(z)=-\log\left(\frac{z}{2}\right)b_{er}(z)+\frac{\pi}{2}b_{ei}(z)+\sum_{k=0}^{\infty}(-1)^k\frac{\psi(2k+1)}{(2k!)^2}\left[\frac{z^2}{4}\right]^{2k} \tag{42}$$

$$K_{ei}(z)=-\log\left(\frac{z}{2}\right)b_{ei}(z)-\frac{\pi}{4}b_{ei}(z)+\sum_{k=0}^{\infty}(-1)^k\frac{\psi(2k+2)}{((2k+1)!)^2}\left[\frac{z^2}{4}\right]^{2k} \tag{43}$$

Recurrence relations

$$f_{\nu+1}+f_{\nu-1}=-\frac{\nu\sqrt{2}}{z}(f_{\nu}-g_{\nu}) \tag{44a}$$

$$f'_{\nu}=\frac{1}{z\sqrt{2}}\left\{f_{\nu+1}+g_{\nu+1}-f_{\nu-1}-g_{\nu-1}\right\} \tag{44b}$$

$$f'_{\nu}-\frac{\nu}{z}f_{\nu}=\frac{1}{\sqrt{2}}\left\{f_{\nu+1}+g_{\nu+1}\right\} \tag{44c}$$

$$f'_{\nu}+\frac{\nu}{z}f_{\nu}=-\frac{1}{\sqrt{2}}\left\{f_{\nu+1}+g_{\nu+1}\right\} \tag{44d}$$

where, the following combinations are valid

(i) $f_\nu = b_{er_\nu}(z)$ $\quad g_\nu = b_{ei_\nu}(z)$

(ii) $f_\nu = b_{ei_\nu}(z)$ $\quad g_\nu = -b_{er_\nu}(z)$

(iii) $f_\nu = K_{er_\nu}(z)$ $\quad g_\nu = K_{ei_\nu}(z)$

(iv) $f_\nu = K_{ei_\nu}(z)$ $\quad g_\nu = -K_{er_\nu}(z)$

Hence, it can be deduced that:

$$\sqrt{2}b'_{er}(z) = b_{er_1}(z) + b_{ei_1}(z) \tag{45}$$

$$\sqrt{2}b'_{er}(z) = b_{er_1}(z) + b_{ei_1}(z) \tag{46}$$

$$\sqrt{2}k'_{er}(z) = k_{er_1}(z) + k_{ei_1}(z) \tag{47}$$

$$\sqrt{2}k'_{ei}(z) = -k_{er_1}(z) + k_{ei_1}(z) \tag{48}$$

It can be deduced from equation (44) that

$$K'_{er_1}(z) = -\tfrac{1}{z}K_{er1}(z) - \frac{1}{\sqrt{2}}\Big\{K_{er}(z) + K_{ei}(z)\Big\} \tag{49}$$

and

$$K'_{ei_1}(z) = -\tfrac{1}{z}K_{ei1}(z) - \frac{1}{\sqrt{2}}\Big\{-K_{er}(z) + K_{ei}(z)\Big\} \tag{50}$$

Hence, differentiation of (47) gives

$$\sqrt{2}K''_{er}(z) = K'_{er_1}(z) + K'_{ei_1}(z)$$

$$= -\tfrac{1}{z}\Big\{K_{er_1}(z) + K_{ei_1}(z)\Big\} - \sqrt{2}K_{ei}(z)$$

i.e.

$$K''_{er}(z) = \frac{1}{\sqrt{2}z}\Big\{K_{er_1}(z) + K_{ei_1}(z)\Big\} - K_{ei}(z) \tag{51}$$

Similarly, it can be deduced that

$$K''_{ei}(z) = -\frac{1}{\sqrt{2}z}\Big\{-K_{er_1}(z) + K_{ei_1}(z)\Big\} + K_{er}(z) \tag{52}$$

Substituting the solution

$$y=K_{er}(z)+iK_{ei}(z)$$

into

$$\nabla^2 y - i\lambda^2 = 0$$

where

$$\nabla^2 \equiv \left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}\right)$$

then, it can be deduced that:

$$\nabla^2\left\{K_{er}(z)+iK_{ei}(z)\right\}+\lambda^2\left\{-iK_{er}(z)+K_{ei}(z)\right\}=0$$

i.e

$$\nabla^2 K_{er}(z) = -\lambda^2 K_{ei}(z) \tag{53}$$

$$\nabla^2 K_{ei}(z) = \lambda^2 K_{er}(z) \tag{54}$$

where

$$\nabla^2 = \frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}$$

$$z = \lambda x$$

$$\frac{d}{dx} = \lambda\frac{d}{dz}$$

$$\frac{d}{dz} = \left(\frac{1}{\lambda}\right)\frac{d}{dx}$$

Defining

$$\nabla_z^2 = \frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz} \tag{55}$$

then, it can be deduced that:

$$\nabla_z^2 K_{er}(z) = -K_{ei}(z) \tag{56}$$

$$\nabla_z^2 K_{ei}(z) = K_{er}(z) \tag{57}$$

Hence,

$$\frac{d}{dz}\left\{\nabla_z^2 K_{er}(z)\right\} = -K'_{ei}(z)$$

$$= \frac{1}{\sqrt{2}}\left\{-K_{er_1}(z) + K_{ei_1}(z)\right\} \tag{58}$$

and

$$\frac{d}{dz}\left\{\nabla_z^2 K_{ei}(z)\right\} = K'_{er}(z)$$

$$= \frac{1}{\sqrt{2}}\left\{K_{er_1}(z) + K_{ei_1}(z)\right\} \tag{59}$$

and from the solution of equation (30), it can be proved that

$$\nabla^2\nabla^2 K_{er}(z) = \lambda^4 K_{er}(z) \tag{60}$$

and

$$\nabla^2\nabla^2 K_{ei}(z) = \lambda^4 K_{ei}(z) \tag{61}$$

Since there are polynomial approximations for b_{er}, b_{ei}, K_{er}, K_{ei} and their first order derivatives, one may replace b_{er_1}, b_{ei_1}, K_{er_1}, K_{ei_1}, by equations deduced from equations (45)−(48) as follows:

$$b_{er_1}(z) = \frac{1}{\sqrt{2}}\left\{b'_{er}(z) - b'_{ei}(z)\right\} \tag{62}$$

$$b_{ei_1}(z) = \frac{1}{\sqrt{2}}\left\{b'_{er}(z) + b'_{ei}(z)\right\} \tag{63}$$

$$K_{er_1}(z) = \frac{1}{\sqrt{2}}\left\{K'_{er}(z) - K'_{ei}(z)\right\} \tag{64}$$

$$K_{ei_1}(z) = \frac{1}{\sqrt{2}}\left\{K'_{er}(z) + K'_{ei}(z)\right\} \tag{65}$$

Notice also, that equations (51), (52) can be simplified as follows:

$$K''_{er}(z) = -\frac{1}{2}K'_{er}(z) - K_{ei}(z) \tag{66}$$

$$K''_{ei}(z) = -\frac{1}{2}K'_{ei}(z) + K_{er}(z) \tag{67}$$

which can also be obtained as a direct result of equations (56) and (57).

APPENDIX D

APPENDIX D

Analysis of singular integrals of $\underset{\sim}{G}$ and $\underset{\sim}{T}$ parameters for thin plates on elastic foundations

1–Introductory concepts

From the previous analysis given in Appendix C, it was shown that

$$\lim_{z \to 0} K_{ei}(z) = -\frac{\pi}{4}$$

$$\lim_{z \to 0} K'_{ei}(z) = 0$$

$$\lim_{z \to 0} \left\{ K'_{er}(z) + \frac{1}{z} \right\} = 0$$

$$\lim_{z \to 0} \left\{ K_{er}(z) + \log z \right\} = \log z - \gamma$$

Hence, the results in the following table can be proved

f(z)	$\lim_{z \to 0} f(z)$
K_{ei}	$-\frac{\pi}{4}$
K'_{ei}	0
$K_{er}(z)+\log(z)$	$\log 2 - \gamma$
$K'_{er}(z)+\frac{1}{z}$	0
$A(z)=K_{er}(z)-\frac{2}{z}K'_{ei}(z)$	$-\frac{1}{2}$
$A'(z)=K'_{er}(z)-\frac{2}{z}A(z)$	0
$B(z)+\frac{2}{z^2}=K_{ei}(z)+\frac{2}{z}K'_{er}(z)+\frac{2}{z^2}$	0
$\frac{K'_{er}(z)}{z}+\frac{\log(z)}{2}$	$\frac{1}{2}\left\{\log 2 - \gamma + \frac{1}{2}\right\}$

f(z)	$\lim_{z\to 0} f(z)$
$\frac{K'_{er}(z)}{z}+\frac{1}{z^2}$	$\frac{\pi}{8}$
$K''_{er}(z)-\frac{1}{z^2}$	$\frac{\pi}{8}$
$K''_{ei}(z)+\frac{1}{2}\log(z)$	$\frac{1}{2}\left\{\log 2-\gamma-\frac{1}{2}\right\}$

2–Singular integral terms

They are developed when the source point is on the element, in terms like

$$\int_{element\ i} f(x-x_i,y-y_i)ds$$

Notice that, on that element we have

$$\frac{\partial r}{\partial n}=0$$

$$\frac{\partial r}{\partial t}=\pm 1$$

Basic theorems

Let f(r), it is assumed that it dose not contain terms like $\frac{\partial r}{\partial n}, \frac{\partial r}{\partial t}, \frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial n_1}, \frac{\partial r}{\partial n_o}$

1– $$\int_{(x_1,y_1)}^{(x_2,y_2)} f(r)\ ds = 2\int_0^{R_o} f(r)\ dr$$

where

$$R_o=\frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$=\sqrt{(x_2-x_i)^2+(y_2-y_i)^2}$$

$$r=\sqrt{(x-x_i)^2+(y-y_i)^2}$$

2– $$\int_{(x_1,y_1)}^{(x_2,y_2)} f(r)\,\frac{\partial r}{\partial n_o}\,ds = 0$$

3– $$\int_{(x_1,y_1)}^{(x_2,y_2)} f(r)\,\frac{\partial r}{\partial x}\,\frac{\partial r}{\partial y}\,ds = 2(\hat{\imath}\cdot\hat{\imath})(\hat{\imath}\cdot\hat{\jmath})\int_0^{R_o} f(r)\,dr$$

Results

If we let $z=\lambda r$, $dz=\lambda dr$, $z_o=\lambda R_o$, we have the following.

i– $$\int_{(x_1,y_1)}^{(x_2,y_2)} f(z)\,ds = \frac{2}{\lambda}\int_0^{z_o} f(z)\,dz$$

ii– $$\int_{(x_1,y_1)}^{(x_2,y_2)} f(z)\,\frac{\partial r}{\partial n_o}\,ds = 0$$

where $\frac{\partial r}{\partial n_o} \equiv \frac{\partial r}{\partial n}, \frac{\partial r}{\partial t}, \frac{\partial r}{\partial x}, \cdots$etc

iii– $$\int_{(x_1,y_1)}^{(x_2,y_2)} f(z)\,\frac{\partial r}{\partial x}\,\frac{\partial r}{\partial y}\,ds = \frac{2}{\lambda}(\hat{\imath}\cdot\hat{\imath})(\hat{\imath}\cdot\hat{\jmath})\int_0^{z_o} f(z)\,dz$$

Notice that

$$\hat{n}=l\,\hat{\imath} + m\,\hat{\jmath}$$
$$\hat{\imath}=-m\,\hat{\imath} + l\,\hat{\jmath}$$

with

3–Analysis of G^* integrals

3.1 $G^*_{11}=-\frac{1}{2\pi D\lambda^2}K_{ei}(z)$

Since $\lim_{z\to 0} K_{ei}(z)=\frac{\pi}{4}$, the above term is not singular and the following integrals

$$\int_{element\ i} G^*_{11}\ ds$$

can be dealt with using Gaussian quadrature as usual

3.2 $\underline{G^*_{12}, G^*_{21}, G^*_{13}\ G^*_{31}}$

According to previous theorems, it can be deduced that, on the element which contains the source point we have the following results:

$$\int_i G^*_{12}\ ds=0$$

$$\int_i G^*_{21}\ ds=0$$

$$\int_i G^*_{13}\ ds=0$$

$$\int_i G^*_{31}\ ds=0$$

3.3 $\underline{G^*_{33}, G^*_{23}}$

Considering the following,

$$f(z)=\frac{\hat{n}_o\cdot\hat{n}}{z}K'_{ei}(z)+\frac{\partial r}{\partial n}\ \frac{\partial r}{\partial n_o}\ A(z)$$

then

$$\int_i f(z)\ ds\ =(\hat{n}_o\cdot\hat{n})\int_i \frac{K'_{ei}(z)}{z}\ ds$$

and

$$\int_i \frac{K'_{ei}(z)}{z}\ ds=\int_i \left\{\frac{K'_{ei}(z)}{z}\ +\ 0.5\ \log\ (z)\right\}ds-\int 0.5\ \log(z)\ ds$$

Let

$$K_{eim}(z)=\frac{K'_{ei}(z)}{z}\ +0.5\ \log(z)$$

$$\lim_{z\rightarrow 0} K_{eim}(z)=\ \frac{1}{2}(\log(2)-\gamma+\frac{1}{2})$$

i.e

$$\int_i K_{eim}(z)\ dz$$

can be found by Gaussian quadrature. And

$$\int \log(z)\ ds = \frac{2}{\lambda}\int_0^{z_o} \log(z)\ dz$$

$$= \frac{2}{\lambda}\Big\{z\log(z) - z\Big\}_o^{z_o}$$

$$= \frac{2}{\lambda}\Big\{z_o\log(z_o) - z_o\Big\} = 2R_o\Big\{\log(z_o) - 1\Big\}$$

where

$R_o =$ half the element length,

$z_o = \lambda R_o$

Hence,

$$\int_i \frac{K_{ei}(z)}{z}\ ds = \int_i K_{eim}(z)ds - R_o\Big\{\log(z_o) - 1\Big\}$$

and

$$\int_i f(z)\ ds = (\hat{n}_o \cdot \hat{n})\int_i K_{eim}(z)\ ds - (\hat{n}_o \cdot \hat{n})R_o\Big\{\log(z_o) - 1\Big\}$$

Hence it can be deduced that:

a) for $G^*_{22} = -\frac{1}{2\pi D}\Big\{\frac{\hat{n}_1 \cdot \hat{n}}{z}K'_{ei}(z) + \frac{\partial r}{\partial x}\frac{\partial r}{\partial n}A(z)\Big\}$

$$\int_i G^*_{22}ds = -\Big(\frac{\hat{i} \cdot \hat{n}}{2\pi D}\Big)\int_i K_{eim}(z)\ ds + \Big(\frac{\hat{i} \cdot \hat{n}}{2\pi D}\Big)R_o\Big\{\log(z_o) - 1\Big\}$$

b) Similarly,

$$\int_i G^*_{23}ds = -\Big(\frac{\hat{j} \cdot \hat{n}}{2\pi D}\Big)\int_i K_{eim}(z)\ ds + \Big(\frac{\hat{j} \cdot \hat{n}}{2\pi D}\Big)R_o\Big\{\log(z_o) - 1\Big\}$$

where

$$K_{eim}(z)=\frac{K'_{ei}(z)}{z}+0.5\log(z)$$

3.4 G^*_{32}, G^*_{33}

Let $f(z)=\frac{\hat{n}_o\cdot\hat{\imath}}{z}K'_{ei}(z)+\frac{\partial r}{\partial t}\frac{\partial r}{\partial n_o}A(z)$

From previous theorems, we can deduce that:

$$\int_i f(z)\,ds=\int_i\Big\{(\hat{n}_o\cdot\hat{\imath})\,K_{eim}(z)+\frac{\partial r}{\partial t}\frac{\partial r}{\partial n_o}A(z)\Big\}ds-(\hat{n}_o\cdot\hat{\imath})R_o\Big\{\log(z_o)-1\Big\}$$

and the first term may be evaluated by means of Gaussian quadrature

$$\int_i G^*_{32}ds=-\frac{1}{2\pi D}\int_i\Big\{(\hat{\imath}\cdot\hat{\imath})K_{eim}(z)+\frac{\partial r}{\partial t}\frac{\partial r}{\partial x}A(z)\Big\}ds+\Big(\frac{\hat{\imath}\cdot\hat{\imath}}{2\pi D}\Big)R_o\Big\{\log(z_o)-1\Big\}$$

$$\int_i G^*_{33}ds=-\frac{1}{2\pi D}\int_i\Big\{(\hat{\jmath}\cdot\hat{\imath})K_{eim}(z)+\frac{\partial r}{\partial t}\frac{\partial r}{\partial y}A(z)\Big\}ds+\Big(\frac{\hat{\jmath}\cdot\hat{\imath}}{2\pi D}\Big)R_o\Big\{\log(z_o)-1\Big\}$$

4–Analysis of $\underline{T}^*$ integrals

4.1 $T^*_{11}=\frac{\lambda}{2\pi}\frac{\partial r}{\partial n}K'_{er}(z)$

$$\int_i T^*_{11}\,ds=0$$

4.2 T^*_{12}, T^*_{13}

From previous analysis, we can deduce the following

$$\int_i T^*_{12}\,ds=\frac{\lambda^2 l}{2\pi}\int_i\frac{K'_{er}(z)}{z}\,ds$$

$$\int_i T^*_{13}\,ds=\frac{\lambda^2 m}{2\pi}\int_i\frac{K'_{er}(z)}{z}\,ds$$

$$\int_i\frac{K'_{er}(z)}{z}\,ds=\frac{2}{\lambda}\int_0^{z_o}\frac{K'_{er}(z)}{z}\,dz=\frac{2}{\lambda}\Big\{\frac{K_{er}(z)}{z}+\int_0^{z_o}\frac{K_{er}(z)}{z^2}\,dz\Big\}$$

or

$$\int_{i} \frac{K'_{er}(z)}{z}\, ds = \int_{0}^{z_o} \left\{ \frac{K'_{er}(z)}{z} + \frac{1}{z^2} \right\} dz - \frac{2}{\lambda} \int_{0}^{z_o} \frac{1}{z^2}\, dz$$

Defining

$$K_{erm}(z) = \frac{K'_{er}(z)}{z} + \frac{1}{z^2}$$

then

$$\lim_{z \to 0} K_{erm}(z) = \frac{\pi}{8}$$

and

$$\int_{z} K_{erm}(z)\, ds$$

can be found by means of Gaussian quadrature; however,

$$-\frac{2}{\lambda} \int_{0}^{z_o} \frac{1}{z^2}\, dz = \frac{2}{\lambda z_o} - \frac{2}{\lambda} \lim_{\epsilon \to 0} \frac{1}{\epsilon}$$

which is a divergent integral

4.3 $\underline{T^*_{21}}$

$$\int_{i} T^*_{21}\, ds = \frac{1}{4\pi} \int_{i} \left\{ (1+\nu) K_{er}(z) - (1-\nu) A(z) \right\} ds$$

$$= \frac{1}{4\pi} \left\{ \int_{i} \left\{ (1+\nu)\{K_{er}(z) + \log(z)\} - (1-\nu) A(z) \right\} ds - \frac{(1+\nu)}{4\pi} \int_{i} \log(z)\, ds \right\}$$

let

$$\left(T^*_{21}\right)_m = \frac{1}{4\pi} \left\{ (1+\nu)\{K_{er}(z) + \log(z)\} - (1-\nu) A(z) \right\}$$

then

$$\int_{i} T^*_{21} ds = \int_{i} \left(T^*_{21}\right)_m dz - \left(\frac{1+\nu}{4\pi}\right) 2R_o \log(z_o)$$

$(T^*_{21})_m$ is regular and its integration can be carried out by means of Gaussian quadrature.

4.4 T^*_{22}, T^*_{23}, T^*_{31}, T^*_{32}, T^*_{33}

$$\int_i T^*_{22}\ ds = 0$$

$$\int_i T^*_{23}\ ds = 0$$

$$\int_i T^*_{31}\ ds = 0$$

$$\int_i T^*_{32}\ ds = 0$$

$$\int_i T^*_{33}\ ds = 0$$

APPENDIX E

APPENDIX E

FUNDAMENTAL SOLUTION KERNELS FOR THIN PLATES

based on $w_1^* = \dfrac{r^2(\log r - 1)}{8\pi D}$

$$U_{11}^* = w_1^* = \frac{r^2(\log r - 1)}{8\pi D}$$

$$U_{12}^* = \frac{\partial w^*}{\partial x} = \frac{r}{8\pi D}(2\log r - 1)\frac{\partial r}{\partial x}$$

$$U_{13}^* = \frac{\partial w^*}{\partial y} = \frac{r}{8\pi D}(2\log r - 1)\frac{\partial r}{\partial y}$$

$$U_{21}^* = \frac{r}{8\pi D}(2\log r - 1)\frac{\partial r}{\partial n}$$

$$U_{22}^* = \frac{1}{8\pi D}\left\{(\hat{n}\cdot\hat{j})(2\log r - 1) + 2\frac{\partial r}{\partial n}\frac{\partial r}{\partial x}\right\}$$

$$U_{23}^* = \frac{1}{8\pi D}\left\{(\hat{n}\cdot\hat{j})(2\log r - 1) + 2\frac{\partial r}{\partial n}\frac{\partial r}{\partial y}\right\}$$

$$U_{31}^* = \frac{r}{8\pi D}(2\log r - 1)\frac{\partial r}{\partial t}$$

$$U_{32}^* = \frac{1}{8\pi D}\left\{(\hat{i}\cdot\hat{i})(2\log r - 1) + 2\frac{\partial r}{\partial t}\frac{\partial r}{\partial x}\right\}$$

$$U_{33}^* = \frac{1}{8\pi D}\left\{(\hat{i}\cdot\hat{j})(2\log r - 1) + 2\frac{\partial r}{\partial t}\frac{\partial r}{\partial y}\right\}$$

$$T_{11}^* = (Q_n)_1 = -D\frac{\partial}{\partial n}(\nabla^2 w_1^*) = -\frac{1}{2\pi r}\frac{\partial r}{\partial n}$$

$$T_{12}^* = (Q_n)_2 = -\frac{1}{2\pi r^2}\left\{(\hat{i}\cdot\hat{n}) - 2\frac{\partial r}{\partial n}\cdot\frac{\partial r}{\partial x}\right\}$$

$$T_{13}^* = (Q_n)_3 = -\frac{1}{2\pi r^2}\left\{(\hat{j}\cdot\hat{n}) - 2\frac{\partial r}{\partial n}\cdot\frac{\partial r}{\partial y}\right\}$$

$$T_{21}^* = (M_n)_1 = \frac{1}{8\pi}\left\{2 + (1+\nu)\log r + (1+\nu)\left[2\left(\frac{\partial r}{\partial n}\right)^2 - 1\right]\right\}$$

$$T_{22}^* = (M_n)_2 = -\frac{1}{4\pi r}\left\{(1+\nu)\frac{\partial r}{\partial x} + 2(1-\nu)\frac{\partial r}{\partial n}\left\{(\hat{n}\cdot\hat{i}) - \frac{\partial r}{\partial n}\cdot\frac{\partial r}{\partial x}\right\}\right\}$$

$$T^*_{23}=(M_n)_3=-\frac{1}{4\pi r}\left\{(1+\nu)\frac{\partial r}{\partial y}+2(1-\nu)\frac{\partial r}{\partial n}\left\{(\hat{n}\cdot\hat{j})-\frac{\partial r}{\partial n}\cdot\frac{\partial r}{\partial y}\right\}\right\}$$

$$T^*_{31}=(M_{tn})_1=-\frac{(1-\nu)}{4\pi}\frac{\partial r}{\partial n}\cdot\frac{\partial r}{\partial t}$$

$$T^*_{32}=(M_{tn})_2=-\frac{(1-\nu)}{4\pi r}\left\{\frac{\partial r}{\partial t}(\hat{\imath}\cdot\hat{n})+2(\hat{\imath}\cdot\hat{\imath})\frac{\partial r}{\partial n}-2\frac{\partial r}{\partial n}\cdot\frac{\partial r}{\partial t}\cdot\frac{\partial r}{\partial x}\right\}$$

$$T^*_{33}=(M_{tn})_3=-\frac{(1-\nu)}{4\pi r}\left\{\frac{\partial r}{\partial t}(\hat{\jmath}\cdot\hat{n})+2(\hat{\jmath}\cdot\hat{\imath})\frac{\partial r}{\partial n}-2\frac{\partial r}{\partial n}\cdot\frac{\partial r}{\partial t}\cdot\frac{\partial r}{\partial y}\right\}$$

Loading terms:

$$T_i=\oint \rho p^*_i\ ds-\oint\frac{\partial P}{\partial n}Q^*_i\ ds$$

Let "f" be defined such that:

$$w^*=\nabla^2 f$$

$$f=\frac{r^4}{128\pi}(\log r\ -\ 1.5)$$

$$p^*_1=\frac{\partial f}{\partial n}=\frac{r^3}{128\pi}(4\ \log r\ -5)\frac{\partial r}{\partial n}$$

$$p^*_2=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial n}\right)=\frac{r^3}{64\pi}\left\{(\hat{n}\cdot\hat{\imath})(2\ \log r\ -2.5)+(4\ \log r\ -3)\frac{\partial r}{\partial n}\frac{\partial r}{\partial x}\right\}$$

$$p^*_3=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial n}\right)=\frac{r^3}{64\pi}\left\{(\hat{n}\cdot\hat{\jmath})(2\ \log r\ -2.5)+(4\ \log r\ -3)\frac{\partial r}{\partial n}\frac{\partial r}{\partial y}\right\}$$

$$Q^*_1=f=\frac{r^4}{128\pi}(\log r\ -1.5)$$

$$Q^*_2=\frac{\partial f}{\partial x}=\frac{r^3}{128\pi}(4\ \log r\ -5)\frac{\partial r}{\partial x}$$

$$Q^*_3=\frac{\partial f}{\partial y}=\frac{r^3}{128\pi}(4\ \log r\ -5)\frac{\partial r}{\partial y}$$

APPENDIX F

APPENDIX F

DERIVATION OF THE $\underline{h}$ MATRIX

$$\begin{bmatrix} M_n \\ M_{tn} \\ Q_n \end{bmatrix} = \underline{h} \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}$$

i.e

$$\underline{h} = D \begin{bmatrix} \frac{\partial}{\partial n} & \nu\frac{\partial}{\partial t} & 0 \\ \frac{(1-\nu)}{2}\frac{\partial}{\partial t} & \frac{(1-\nu)}{2}\frac{\partial}{\partial n} & 0 \\ \frac{(1-\nu)}{2}\lambda^2 & 0 & \frac{(1-\nu)}{2}\lambda^2\frac{\partial}{\partial n} \end{bmatrix} \underline{g}$$

$$=\beta_1\underline{\bar{h}}(\lambda_1)+\beta_2\underline{\bar{h}}(\lambda_2)+\beta_3\underline{\bar{h}}(\lambda_3)$$

where

$$\beta_s = D\alpha_s$$

i.e

$$\beta_1 = \frac{1}{2\pi(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}$$

$$\beta_2 = \frac{1}{2\pi(\lambda_2^2-\lambda_1^2)(\lambda_2^2-\lambda_3^2)}$$

$$\beta_3 = \frac{1}{2\pi(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)}$$

and

$$\underline{\bar{h}}(\lambda_s) = D\begin{bmatrix} \frac{\partial}{\partial n} & \nu\frac{\partial}{\partial t} & 0 \\ \frac{(1-\nu)}{2}\frac{\partial}{\partial t} & \frac{(1-\nu)}{2}\frac{\partial}{\partial n} & 0 \\ \frac{(1-\nu)}{2}\lambda^2 & 0 & \frac{(1-\nu)}{2}\lambda^2\frac{\partial}{\partial n} \end{bmatrix}\underline{\bar{g}}(\lambda_s)$$

$$=\underline{E}(\lambda_s)K_o(\lambda_s r)+\underline{F}(\lambda_s)K_1(\lambda_s r)$$

$$\bar{h}_{11}(c)=\frac{\partial}{\partial n}\bar{g}_{11}(c)+\nu\frac{\partial}{\partial t}\bar{g}_{21}(c)$$

$$\bar{h}_{12}(c)=\frac{\partial}{\partial n}\bar{g}_{12}(c)+\nu\frac{\partial}{\partial t}\bar{g}_{22}(c)$$

$$\bar{h}_{13}(c)=\frac{\partial}{\partial n}\bar{g}_{13}(c)+\nu\frac{\partial}{\partial t}\bar{g}_{23}(c)$$

$$\bar{h}_{21}(c)=\frac{(1-\nu)}{2}\left\{\frac{\partial}{\partial t}\bar{g}_{11}(c)+\nu\frac{\partial}{\partial n}\bar{g}_{21}(c)\right\}$$

$$\bar{h}_{22}(c)=\frac{(1-\nu)}{2}\left\{\frac{\partial}{\partial t}\bar{g}_{12}(c)+\nu\frac{\partial}{\partial n}\bar{g}_{22}(c)\right\}$$

$$\bar{h}_{23}(c)=\frac{(1-\nu)}{2}\left\{\frac{\partial}{\partial t}\bar{g}_{13}(c)+\nu\frac{\partial}{\partial n}\bar{g}_{23}(c)\right\}$$

$$\bar{h}_{31}(c)=\frac{(1-\nu)}{2}\lambda^2\left\{\bar{g}_{11}(c)+\nu\frac{\partial}{\partial n}\bar{g}_{21}(c)\right\}$$

$$\bar{h}_{32}(c)=\frac{(1-\nu)}{2}\lambda^2\left\{\bar{g}_{12}(c)+\nu\frac{\partial}{\partial n}\bar{g}_{22}(c)\right\}$$

$$\bar{h}_{33}(c)=\frac{(1-\nu)}{2}\lambda^2\left\{\bar{g}_{13}(c)+\nu\frac{\partial}{\partial n}\bar{g}_{23}(c)\right\}$$

Hence, it can be proved that:

$$\underline{\bar{h}}(c)\left\{\underline{E}'(c)+\underline{E}''(c)\right\}K_o(cr)+\left\{\underline{F}'(c)+\underline{F}''(c)\right\}K_1(cr)$$

$$E'_{11}=-c\left\{\frac{\partial r}{\partial n}B'_{11}+\nu\frac{\partial r}{\partial t}B'_{21}\right\}$$

$$E'_{12}=-c\left\{\frac{\partial r}{\partial n}B'_{12}+\nu\frac{\partial r}{\partial t}B'_{22}\right\}$$

$$E'_{13}=-c\left\{\frac{\partial r}{\partial n}B'_{13}+\nu\frac{\partial r}{\partial t}B'_{23}\right\}$$

$$E'_{21}=\frac{-c(1-\nu)}{2}\left\{\frac{\partial r}{\partial t}B'_{11}+\nu\frac{\partial r}{\partial n}B'_{21}\right\}$$

$$E'_{22}=\frac{-c(1-\nu)}{2}\left\{\frac{\partial r}{\partial t}B'_{12}+\nu\frac{\partial r}{\partial n}B'_{22}\right\}$$

$$E'_{23}=\frac{-c(1-\nu)}{2}\left\{\frac{\partial r}{\partial t}B'_{13}+\nu\frac{\partial r}{\partial n}B'_{23}\right\}$$

$$E'_{31}=\frac{(1-\nu)}{2}\lambda^2\left\{A'_{11}-c\frac{\partial r}{\partial n}B'_{31}\right\}$$

$$E'_{32}=\frac{(1-\nu)}{2}\lambda^2\left\{A'_{12}+\nu\frac{\partial r}{\partial n}B'_{22}\right\}$$

$$E'_{33}=0$$

$$E''_{11}=\frac{\partial A'_{11}}{\partial n}+\nu\frac{\partial A'_{21}}{\partial t}$$

$$E''_{21}=\frac{\partial A'_{12}}{\partial n}+\nu\frac{\partial A'_{22}}{\partial t}$$

$$E''_{13}=0$$

$$E''_{21}=\frac{(1-\nu)}{2}\left\{\frac{\partial A'_{11}}{\partial t}+\frac{\partial A'_{11}}{\partial n}\right\}$$

$$E''_{22}=\frac{(1-\nu)}{2}\left\{\frac{\partial A'_{12}}{\partial t}+\frac{\partial A'_{22}}{\partial n}\right\}$$

$$E''_{23}=0$$

$$E''_{31}=0$$

$$E''_{32}=0$$

$$E''_{33}=\frac{(1-\nu)}{2}\lambda^2\frac{\partial A_{33}}{\partial n}$$

$$F''_{11}=\frac{\partial B'_{11}}{\partial n}+\nu\frac{\partial B'_{21}}{\partial t}$$

$$F''_{12}=\frac{\partial B'_{12}}{\partial n}+\nu\frac{\partial B'_{22}}{\partial t}$$

$$F''_{13}=\frac{\partial B'_{13}}{\partial n}+\nu\frac{\partial B'_{23}}{\partial t}$$

$$F''_{21}=\frac{(1-\nu)}{2}\left\{\frac{\partial B'_{11}}{\partial t}+\frac{\partial B'_{21}}{\partial n}\right\}$$

$$F''_{22}=\frac{(1-\nu)}{2}\left\{\frac{\partial B'_{12}}{\partial t}+\frac{\partial B'_{22}}{\partial n}\right\}$$

$$F''_{23}=\frac{(1-\nu)}{2}\left\{\frac{\partial B'_{13}}{\partial t}+\frac{\partial B'_{23}}{\partial n}\right\}$$

$$F''_{31}=\frac{(1-\nu)}{2}\lambda^2\frac{\partial B_{31}}{\partial n}$$

$$F''_{32}=\frac{(1-\nu)}{2}\lambda^2\frac{\partial B_{32}}{\partial n}$$

$$F''_{33}=0$$

$$F'_{11}=-\left\{\frac{\partial r}{\partial n}[cA'_{11}+\frac{B'_{11}}{r}]+\nu\frac{\partial r}{\partial t}[cA'_{21}+\frac{B'_{21}}{r}]\right\}$$

$$F'_{12}=-\left\{\frac{\partial r}{\partial n}[cA'_{12}+\frac{B'_{12}}{r}]+\nu\frac{\partial r}{\partial t}[cA'_{22}+\frac{B'_{22}}{r}]\right\}$$

$$F'_{13}=-\left\{\frac{\partial r}{\partial n}(\frac{B'_{13}}{r})+\nu\frac{\partial r}{\partial t}(\frac{B'_{23}}{r})\right\}$$

$$F'_{21}=\frac{-(1-\nu)}{2}\left\{\frac{\partial r}{\partial t}[cA'_{11}+\frac{B'_{11}}{r}]+\frac{\partial r}{\partial n}[cA'_{21}+\frac{B'_{21}}{r}]\right\}$$

$$F'_{22}=\frac{-(1-\nu)}{2}\left\{\frac{\partial r}{\partial t}[cA'_{12}+\frac{B'_{12}}{r}]+\frac{\partial r}{\partial n}[cA'_{22}+\frac{B'_{22}}{r}]\right\}$$

$$F'_{23}=\frac{-(1-\nu)}{2}\left\{\frac{\partial r}{\partial t}\frac{B'_{13}}{r}+\frac{\partial r}{\partial n}\frac{B'_{23}}{r}\right\}$$

$$F'_{31}=\frac{(1-\nu)}{2}\lambda^2\left\{B'_{11}-\frac{\partial r}{\partial n}(\frac{B'_{31}}{r})\right\}$$

$$F'_{32}=\frac{(1-\nu)}{2}\lambda^2\left\{B'_{12}-\frac{\partial r}{\partial n}\frac{B'_{32}}{r}\right\}$$

$$F'_{33}=\frac{(1-\nu)}{2}\lambda^2\left\{B'_{13}-c\frac{\partial r}{\partial n}A'_{33}\right\}$$

with $n_o \equiv n$ or t

$$\frac{\partial A'_{11}}{\partial n_o} = l\,\frac{\partial A_{11}}{\partial n_o} + m\,\frac{\partial A_{21}}{\partial n_o} \qquad \frac{\partial B_{11}}{\partial n_o} = l\,\frac{\partial B_{11}}{\partial n_o} + m\,\frac{\partial B_{21}}{\partial n_o}$$

$$\frac{\partial A'_{21}}{\partial n_o} = l\,\frac{\partial A_{12}}{\partial n_o} + m\,\frac{\partial A_{22}}{\partial n_o} \qquad \frac{\partial B'_{12}}{\partial n_o} = l\,\frac{\partial B_{12}}{\partial n_o} + m\,\frac{\partial B_{22}}{\partial n_o}$$

$$\frac{\partial A'_{13}}{\partial n_o}=0 \qquad \frac{\partial B'_{13}}{\partial n_o} = l\,\frac{\partial B_{13}}{\partial n_o} + m\,\frac{\partial B_{23}}{\partial n_o}$$

$$\frac{\partial A'_{21}}{\partial n_o} = -m\,\frac{\partial A_{11}}{\partial n_o} + l\,\frac{\partial A_{21}}{\partial n_o} \qquad \frac{\partial B'_{21}}{\partial n_o} = -m\,\frac{\partial B_{11}}{\partial n_o} + l\,\frac{\partial B_{21}}{\partial n_o}$$

$$\frac{\partial A'_{22}}{\partial n_o} = -m\,\frac{\partial A_{12}}{\partial n_o} + l\,\frac{\partial A_{22}}{\partial n_o} \qquad \frac{\partial B'_{22}}{\partial n_o} = -m\,\frac{\partial B_{12}}{\partial n_o} + l\,\frac{\partial B_{22}}{\partial n_o}$$

$$\frac{\partial A'_{23}}{\partial n_o}=0 \qquad \frac{\partial B'_{23}}{\partial n_o} = -m\,\frac{\partial B_{13}}{\partial n_o} + l\,\frac{\partial B_{23}}{\partial n_o}$$

$$\frac{\partial A'_{31}}{\partial n_o}=0 \qquad \frac{\partial B'_{31}}{\partial n_o}=\frac{\partial B_{31}}{\partial n_o}$$

$$\frac{\partial A'_{32}}{\partial n_o}=0 \qquad \frac{\partial B'_{32}}{\partial n_o}=\frac{\partial B_{32}}{\partial n_o}$$

$$\frac{\partial A_{33}'}{\partial n_o}=\frac{\partial A_{33}}{\partial n_o} \qquad \frac{\partial B_{33}'}{\partial n_o}=0$$

Using the following theorems:

(i) $$\frac{\partial^2 r}{\partial n_1 \partial n_2}=\frac{1}{r}\left\{(\hat{n}_1\cdot\hat{n}_2)-\frac{\partial r}{\partial n_1}\frac{\partial r}{\partial n_2}\right\}$$

(ii) $$\frac{\partial}{\partial n_3}\left(\frac{\partial r}{\partial n_1}\frac{\partial r}{\partial n_2}\right)=\frac{1}{r}\left\{(\hat{n}_2\cdot\hat{n}_3)\frac{\partial r}{\partial n_1}+(\hat{n}_3\cdot\hat{n}_1)\frac{\partial r}{\partial n_2}-2\frac{\partial r}{\partial n_1}\frac{\partial r}{\partial n_2}\frac{\partial r}{\partial n_3}\right\}$$

(iii) $$\frac{\partial}{\partial n_2}\left(\frac{1}{r}\frac{\partial r}{\partial n_1}\right)=\frac{1}{r^2}\left\{(\hat{n}_1\cdot\hat{n}_2)-2\frac{\partial r}{\partial n_1}\frac{\partial r}{\partial n_2}\right\}$$

(iv) $$\frac{\partial}{\partial n_3}\left(\frac{1}{r}\frac{\partial r}{\partial n_1}\frac{\partial r}{\partial n_2}\right)=\frac{1}{r^2}\left\{(\hat{n}_2\cdot\hat{n}_3)\frac{\partial r}{\partial n_1}+(\hat{n}_3\cdot\hat{n}_1)\frac{\partial r}{\partial n_2}-3\frac{\partial r}{\partial n_1}\frac{\partial r}{\partial n_2}\frac{\partial r}{\partial n_3}\right\}$$

it can be shown that

$$\frac{\partial A_{11}}{\partial n}=\frac{2c^2}{r}\left\{\alpha(c^2-\beta)+\gamma\right\}\frac{\partial r}{\partial y}\left\{m-\frac{\partial r}{\partial y}\frac{\partial r}{\partial n}\right\}$$

$$\frac{\partial A_{11}}{\partial t}=\frac{2c^2}{r}\left\{\alpha(c^2-\beta)+\gamma\right\}\frac{\partial r}{\partial y}\left\{l-\frac{\partial r}{\partial y}\frac{\partial r}{\partial t}\right\}$$

$$\frac{\partial A_{12}}{\partial n}=-\frac{\alpha c^2}{r}\left\{c^2-\beta+\frac{\gamma}{\alpha}\right\}\left\{m\frac{\partial r}{\partial x}+l\frac{\partial r}{\partial y}-2\frac{\partial r}{\partial x}\frac{\partial r}{\partial y}\frac{\partial r}{\partial n}\right\}$$

$$\frac{\partial A_{12}}{\partial t}=-\frac{\alpha c^2}{r}\left\{c^2-\beta+\frac{\gamma}{\alpha}\right\}\left\{l\frac{\partial r}{\partial x}-m\frac{\partial r}{\partial y}-2\frac{\partial r}{\partial x}\frac{\partial r}{\partial y}\frac{\partial r}{\partial n}\right\}$$

$$\frac{\partial A_{13}}{\partial n}=0$$

$$\frac{\partial A_{13}}{\partial t}=0$$

$$\frac{\partial A_{21}}{\partial n}=\frac{\partial A_{12}}{\partial n}$$

$$\frac{\partial A_{21}}{\partial t}=\frac{\partial A_{12}}{\partial t}$$

$$\frac{\partial A_{22}}{\partial n}=\frac{2c^2}{r}\left\{\alpha(c^2-\beta)+\gamma\right\}\frac{\partial r}{\partial x}\left\{l-\frac{\partial r}{\partial x}\frac{\partial r}{\partial n}\right\}$$

$$\frac{\partial A_{22}}{\partial t}=\frac{2c^2}{r}\left\{\alpha(c^2-\beta)+\gamma\right\}\frac{\partial r}{\partial x}\left\{-m-\frac{\partial r}{\partial x}\frac{\partial r}{\partial t}\right\}$$

$$\frac{\partial A_{23}}{\partial n}=0$$

$$\frac{\partial A_{23}}{\partial t}=0$$

$$\frac{\partial A_{31}}{\partial n}=\frac{\partial A_{32}}{\partial n}=\frac{\partial A_{33}}{\partial n}=0$$

$$\frac{\partial A_{31}}{\partial t}=\frac{\partial A_{32}}{\partial t}=\frac{\partial A_{33}}{\partial t}=0$$

$$\frac{\partial B_{11}}{\partial n}=\frac{c}{r^2}\left\{\alpha(c^2-\beta)+\gamma\right\}\left\{4m\frac{\partial r}{\partial y}+\frac{\partial r}{\partial n}-6\frac{\partial r}{\partial n}\left(\frac{\partial r}{\partial y}\right)^2\right\}$$

$$\frac{\partial B_{11}}{\partial t}=\frac{c}{r^2}\left\{\alpha(c^2-\beta)+\gamma\right\}\left\{4l\frac{\partial r}{\partial y}+\frac{\partial r}{\partial t}-6\frac{\partial r}{\partial t}\left(\frac{\partial r}{\partial y}\right)^2\right\}$$

$$\frac{\partial B_{12}}{\partial n}=-\frac{2\alpha c}{r^2}\left\{c^2-\beta+\frac{\gamma}{\alpha}\right\}\left\{m\frac{\partial r}{\partial x}+l\frac{\partial r}{\partial y}-3\frac{\partial r}{\partial x}\frac{\partial r}{\partial y}\frac{\partial r}{\partial n}\right\}$$

$$\frac{\partial B_{12}}{\partial t}=-\frac{2\alpha c}{r^2}\left\{c^2-\beta+\frac{\gamma}{\alpha}\right\}\left\{l\frac{\partial r}{\partial x}-m\frac{\partial r}{\partial y}-3\frac{\partial r}{\partial x}\frac{\partial r}{\partial y}\frac{\partial r}{\partial t}\right\}$$

$$\frac{\partial B_{13}}{\partial n}=-\frac{c}{r}\left\{c^2-\lambda^2\right\}\left\{l-\frac{\partial r}{\partial x}\frac{\partial r}{\partial n}\right\}$$

$$\frac{\partial B_{13}}{\partial t}=-c(c^2-\lambda^2)(-m-\frac{\partial r}{\partial x}\cdot\frac{\partial r}{\partial t})$$

$$\frac{\partial B_{21}}{\partial n}=\frac{\partial B_{12}}{\partial n}$$

$$\frac{\partial B_{21}}{\partial t}=\frac{\partial B_{12}}{\partial t}$$

$$\frac{\partial B_{22}}{\partial n}=\frac{c}{r^2}\left\{\alpha(c^2-\beta)+\gamma\right\}\left\{4l\frac{\partial r}{\partial x}+\frac{\partial r}{\partial n}-6\frac{\partial r}{\partial n}\left(\frac{\partial r}{\partial x}\right)^2\right\}$$

$$\frac{\partial B_{22}}{\partial t}=\frac{c}{r^2}\left\{\alpha(c^2-\beta)+\gamma\right\}\left\{-4m\frac{\partial r}{\partial x}+\frac{\partial r}{\partial t}-6\frac{\partial r}{\partial t}\left(\frac{\partial r}{\partial x}\right)^2\right\}$$

$$\frac{\partial B_{23}}{\partial n}=-c(c^2-\lambda^2)(m-\frac{\partial r}{\partial y}\cdot\frac{\partial r}{\partial n})$$

$$\frac{\partial B_{23}}{\partial t}=-c(c^2-\lambda^2)(l-\frac{\partial r}{\partial y}\cdot\frac{\partial r}{\partial t})$$

$$\frac{\partial B_{31}}{\partial n}=\frac{c\gamma}{r\lambda^2}\left\{c^2-\lambda^2\right\}\left\{l-\frac{\partial r}{\partial x}\frac{\partial r}{\partial n}\right\}$$

$$\frac{\partial B_{31}}{\partial t}=\frac{c\gamma}{r\lambda^2}\left\{c^2-\lambda^2\right\}\left\{-m-\frac{\partial r}{\partial x}\frac{\partial r}{\partial t}\right\}$$

$$\frac{\partial B_{32}}{\partial n}=\frac{c\gamma}{r\lambda^2}\left\{c^2-\lambda^2\right\}\left\{m-\frac{\partial r}{\partial y}\frac{\partial r}{\partial n}\right\}$$

$$\frac{\partial B_{32}}{\partial t}=\frac{c\gamma}{r\lambda^2}\left\{c^2-\lambda^2\right\}\left\{l-\frac{\partial r}{\partial y}\frac{\partial r}{\partial t}\right\}$$

$$\frac{\partial B_{33}}{\partial n}=0$$

$$\frac{\partial B_{33}}{\partial t}=0$$

$$\frac{\partial A_{11}(c)}{\partial x}=-\frac{2c^2}{r}\left\{\alpha(c^2-\beta)+\gamma\right\}\frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial y}\right)^2$$

$$\frac{\partial A_{12}(c)}{\partial x}=-\frac{\alpha c^2}{r}\left\{c^2-\beta+\frac{\gamma}{\alpha}\right\}\left\{\frac{\partial r}{\partial y}-2\left(\frac{\partial r}{\partial x}\right)^2\frac{\partial r}{\partial y}\right\}$$

$$\frac{\partial A_{13}(c)}{\partial x}=0$$

$$\frac{\partial A_{21}(c)}{\partial y}=-\frac{\alpha c^2}{r}\left\{c^2-\beta+\frac{\gamma}{\alpha}\right\}\left\{\frac{\partial r}{\partial x}-2\frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial y}\right)^2\right\}$$

$$\frac{\partial A_{22}(c)}{\partial y}=-\frac{2c^2}{r}\left\{\alpha(c^2-\beta)+\gamma\right\}\left(\frac{\partial r}{\partial x}\right)^2\frac{\partial r}{\partial y}$$

$$\frac{\partial A_{23}(c)}{\partial y}=0$$

$$\frac{\partial B_{11}(c)}{\partial x}=\frac{c}{r^2}\left\{\alpha(c^2-\beta)+\gamma\right\}\left\{\frac{\partial r}{\partial x}-6\frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial y}\right)^2\right\}$$

$$\frac{\partial B_{12}(c)}{\partial x}=-\frac{2\alpha c}{r^2}\left\{c^2-\beta+\frac{\gamma}{\alpha}\right\}\left\{\frac{\partial r}{\partial y}-3\left(\frac{\partial r}{\partial x}\right)^2\frac{\partial r}{\partial y}\right\}$$

$$\frac{\partial B_{13}(c)}{\partial x}=-\frac{c}{r}\left\{c^2-\lambda^2\right\}\left\{1-\left(\frac{\partial r}{\partial x}\right)^2\right\}$$

$$\frac{\partial B_{21}(c)}{\partial y}=-\frac{2\alpha c}{r^2}\left\{c^2-\beta+\frac{\gamma}{\alpha}\right\}\left\{\frac{\partial r}{\partial x}-3\frac{\partial r}{\partial x}\left(\frac{\partial r}{\partial y}\right)^2\right\}$$

$$\frac{\partial B_{22}(c)}{\partial y}=\frac{c}{r^2}\left\{\alpha(c^2-\beta)+\gamma\right\}\left\{\frac{\partial r}{\partial y}-6\left(\frac{\partial r}{\partial x}\right)^2\frac{\partial r}{\partial y}\right\}$$

$$\frac{\partial B_{23}(c)}{\partial y}=-\frac{c}{r}\left\{c^2-\lambda^2\right\}\left\{1-\left(\frac{\partial r}{\partial y}\right)^2\right\}$$

APPENDIX G

ANALYTICAL SOLUTION FOR PLATES ON ELASTIC FOUNDATIONS

Appendix G

G-1 ANALYTICAL SOLUTION FOR CIRCULAR THIN PLATES ON ELASTIC FOUNDATIONS

1 – Review of governing equations for Axisymmetric loading

$$(\nabla^4+\kappa^4)w=P/D$$

$$\theta_r=-\frac{dw}{dr}$$

$$Q_r=-D\frac{d}{dr}(\nabla^2 w)$$

$$M_r=-D\left\{\frac{d^2w}{dr^2}+\frac{v}{r}\frac{dw}{dr}\right\}$$

$$=-D\left\{\nabla^2 w+\frac{\nu-1}{r}\frac{dw}{dr}\right\}$$

$$=-D\left\{\nabla^2 w+\frac{D(1-\nu)}{r}\frac{dw}{dr}\right\}$$

$$M_\theta=-D\left\{\nu\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right\}$$

$$=-D\left\{\nu\nabla^2 w+\frac{(1-\nu)}{r}\frac{dw}{dr}\right\}$$

$$=-\nu D\nabla^2 w-\frac{D(1-\nu)}{r}\frac{dw}{dr}$$

$$M_r+M_\theta=-D(1+\nu)\nabla^2 w$$

2 – Solution for uniform P

The general solution of

$$(\nabla^4+\kappa^4)w=P/D$$

is

$$w=w_{pi}+w_{cf}$$

where

$$w_{pi}=P/K$$
$$w_{cf}=\alpha B_{ei}(\kappa r)+\beta B_{er}(\kappa r)$$
α, β are integration constants

Let

$$w_1=B_{ei}(\kappa r), \qquad w_2=B_{er}(\kappa r),$$
$$w_3=P/K$$

$$\theta_1=-\frac{dw_1}{dr}=-\kappa B'_{ei}(\kappa r)$$

$$\theta_2=-\frac{dw_2}{dr}=-\kappa B'_{er}(kr)$$

$$Q_1=-D\kappa^3 B'_{er}(\kappa r), \qquad Q_2=D\kappa^3 B'_{ei}(\kappa r)$$

$$M_1=(M_r)_{\overline{1}}=-D\left[\nabla^2 B_{ei}(\kappa r)\right]+\frac{D(1-\nu)}{r}\frac{d}{dr}(B_{ei})$$

$$=-DK^2 B_{er}(\kappa r)+\frac{DK(1-\nu)}{r}B'_{ei}(\kappa r)$$

$$M_2=-DK^2 b_{ei}(\kappa r)+\frac{DK(1-\nu)}{r}b'_{er}(\kappa r)$$

<u>Notice that</u>

$$\nabla^2 B_{ei}(\kappa r)=\alpha^2 B_{er}(\kappa r)$$
$$\nabla^2 B_{er}(\kappa r)=-K^2 B_{ei}(\kappa r)$$

$$\nabla^2 K_{ei}(\kappa r)=K^2 K_{er}(\kappa r)$$
$$\nabla^2 K_{er}(\kappa r)=-K^2 K_{ei}(\kappa r)$$

where

$$\nabla^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}$$

a) <u>For a simply supported plate</u>

$$w(r_e)=\alpha w_1(r_e)+\beta w_2(r_e)+w_3=0$$

$$M_r(r_e)=\alpha M_1(r_e)+\beta M_2(r_e)=0$$

$$\alpha=\frac{-w_3M_2}{(w_1M_2-w_2M_1)};\qquad \beta=\frac{w_3M_1}{(w_1M_2-w_2M_1)}.$$

b) For a clamped plate

$$\alpha=\frac{-w_3\theta_2}{(w_1\theta_2-w_2\theta_1)};\qquad \beta=\frac{w_3\theta_1}{(w_1\theta_2-w_2\theta_1)}.$$

3- Solution for concentrated force F

$$(\nabla^2+\kappa^4)w=\frac{F}{D}\delta(x-x_o,y-y_o)$$

From fundamental solution report:

$$w_s\equiv w_3=w_{pi}=-\frac{F}{2\pi D\kappa^2}K_{ei}(\kappa r)$$

the corresponding effects:

$$\theta_s\equiv\theta_3=-\frac{dw_3}{dr}=\frac{F}{2\pi D\kappa}K'_{ei}(\kappa r)$$

$$\theta_s\equiv\theta_3 \;=-D\frac{d}{dr}(\nabla^2 w)$$

$$=\frac{FD}{2\pi D\kappa^2}\,\frac{d}{dr}\left[\kappa^2K_{er}(\kappa r)\right]$$

$$=\frac{F\kappa}{2\pi}K'_{er}(kr)$$

$$M_s\equiv M_3=\frac{F}{2\pi}K_{er}(\kappa r)-\frac{(1-\nu)F}{2\pi kr}K'_{ei}(\kappa r)$$

a) For a simply supported plate

$$\alpha=-\left(\frac{w_sM_2-w_2M_s}{w_1M_2-w_2M_1}\right)\qquad \text{evaluated at } r_e$$

$$\beta=-\left(\frac{w_1M_s-w_sM_1}{w_1M_2-w_2M_1}\right)\qquad \text{evaluated at } r_e$$

b) For a clamped plate

$$\alpha = -\left(\frac{w_s\theta_2 - w_2\theta_s}{w_1\theta_2 - w_2\theta_1}\right) \qquad \text{evaluated at } r_e$$

$$\beta = -\left(\frac{w_1\theta_s - w_s\theta_1}{w_1\theta_2 - w_2\theta_1}\right) \qquad \text{evaluated at } r_e$$

c) For a free-free plate

$$\alpha = -\left(\frac{w_s Q_2 - w_2 Q_s}{w_1 Q_2 - w_2 Q_1}\right) \qquad \text{evaluated at } r_e$$

$$\beta = -\left(\frac{w_1 Q_s - w_s Q_1}{w_1 Q_2 - w_2 Q_1}\right) \qquad \text{evaluated at } r_e$$

G-2 ANALYTIC SOLUTION FOR CIRCULAR THICK PLATES ON ELASTIC FOUNDATIONS

1 – Review of governing equations for axisymetric loading

$$D\left[\nabla^4-\frac{(2-\nu)}{(1-\nu)}\frac{\kappa^4}{\lambda^2}\nabla^2+\kappa^4\right]w=-\left[\frac{(2-\nu)}{(1-\nu)}\frac{\nabla^2}{\lambda^2}-1\right]P$$

where

$$\nabla^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\equiv\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)$$

$$\theta_r=\frac{2Q_r}{D(1-\nu)\lambda^2}-\frac{dw}{dr}, \qquad \theta_\theta=0$$

$$Q_r=-D\frac{d}{dr}\left[\nabla^2-\frac{(2-\nu)}{(1-\nu)}\frac{\kappa^4}{\lambda^2}\right]w-\frac{(2-\nu)}{(1-\nu)}\frac{1}{\lambda^2}\frac{dP}{dr}, \qquad Q_\theta=0$$

$$\theta_r=-\frac{2}{(1-\nu)\lambda^2}\frac{d}{dr}\left[\nabla^2-\frac{(2-\nu)}{(1-\nu)}\frac{\kappa^4}{\lambda^2}+\frac{(1-\nu)\lambda^2}{2}\right]w-\frac{(2-\nu)}{(1-\nu)\lambda^2}\frac{dP}{dr}\frac{2}{D(1-\nu)\lambda^2}$$

$$M_r=-D\left[\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}-\frac{\nu}{(1-\nu)}\frac{\kappa^4}{\lambda^2}w\right]+\frac{2}{\lambda^2}\frac{dQ_r}{dr}-\frac{\nu}{(1-\nu)\lambda^2}P$$

$$M_\theta=-D\left[\frac{1}{r}\frac{dw}{dr}+\nu\frac{d^2w}{dr^2}-\frac{\nu}{(1-\nu)\lambda^2}\kappa^4 w\right]+\frac{2}{\lambda^2}\left(\frac{Q_r}{r}\right)-\frac{\nu}{(1-\nu)\lambda^2}P$$

$$M_{r\theta}=0.$$

2–Solution for uniform P

The basic differential equation may be written as follows:

$$(\nabla^4-2b\nabla^2+c)w=P/D$$

or

$$(\nabla^2-\lambda_1^2)(\nabla^2-\lambda_2^2)w=P/D$$

where

$$b=\frac{(2-\nu)}{2(1-\nu)}\frac{\kappa^4}{\lambda^2}; \qquad c=\kappa^4.$$

$$\begin{pmatrix}\lambda_1^2\\ \lambda_2^2\end{pmatrix}=b\pm\sqrt{b^2-c}$$

The general solution is

$$w(r)=\alpha w_1(r)+\beta w_2(r)+w_3$$

where

w_3=particular integral.

$\alpha w_1+\beta_2 w_2$=complementary function.

$w_3=P/K$

and

$$w_1=I_o(\lambda_1 r)$$
$$w_2=I_o(\lambda_2 r)$$

Notice that $w_3$has no effect on dw/dr, Q_r, θ_r and M_r.

<u>Effect of w_1 and w_2</u>

Let $w_i=I_o(\lambda_i r)$

Notice that

$$I_o'(z)=I_1(z)$$

$$I_o''(z)+\frac{1}{z}I_o'(z)=I_o(z)$$

i.e

$$I_1'(z)=I_o(z)-\frac{1}{z}I_1(z)$$

and

$$\nabla^2 I_o(cr)=c^2 I_o(cr)$$

$$\frac{dw_i}{dr}=\lambda_i I_1(\lambda_i r)$$

$$(Q_r)_i = -D\frac{d}{dr}\left[\lambda_i^2 - \frac{(2-\nu)}{(1-\nu)}\frac{\kappa^4}{\lambda^2}\right] I_o(\lambda_i r)$$

$$= -D\lambda_i A_i \cdot I_1(\lambda_i r)$$

where

$$A_i = \lambda_i^2 - \frac{(2-\nu)}{(1-\nu)}\frac{\kappa^4}{\lambda^2}$$

Hence

$$(\theta_r)_i = -\left[\frac{2\lambda_i}{(1-\nu)\lambda^2}A_i + \lambda_i\right] I_1(\lambda_i r)$$

$$= -\lambda_i\left[\frac{2A_i}{(1-\nu)\lambda^2} + 1\right] I_1(\lambda_i r)$$

and

$$(M_r)_i = -D\left[B_i + 2\left(\frac{\lambda_i^2}{\lambda^2}\right)A_i\right] I_o(\lambda_i r) + \frac{D\lambda_i}{r}\left[\frac{2A_i}{\lambda^2} + (1\text{-}\nu)\right] I_1(\lambda_i r)$$

$$B_i = \lambda_i^2 - \frac{\nu}{(1-\nu)}\frac{\kappa^4}{\lambda^2}.$$

<u>Solution for simply-supported plates</u>

At $r = r_e$;

$$w = 0 = \alpha w_1(r_e) + \beta w_2(r_e) + w_3$$
$$M_r = 0 = \alpha M_1(r_e) + \beta M_2(r_e)$$

Hence

$$\alpha = \begin{vmatrix} -w_3 & w_2 \\ 0 & M_2 \end{vmatrix} \Big/ \begin{vmatrix} w_1 & w_2 \\ M_1 & M_2 \end{vmatrix}$$

i.e

$$\alpha = \frac{-w_3 M_2}{(w_1 M_2 - w_2 M_1)}$$

and

$$\beta = \frac{w_3 M_1}{(w_1 M_2 - w_2 M_1)}$$

Solution for clamped plates

At $r=r_e$, $w=0$ and $\theta=0$

i.e

$$w=0=\alpha w_1(r_e)+\beta w_2(r_e)+w_3$$
$$\theta_r=0=\alpha\theta_1(r_e)+\beta\theta_2(r_e)$$

Hence

$$\alpha=\frac{-w_3\theta_2}{(w_1\theta_2-w_2\theta_1)}$$

$$\beta=\frac{w_3\theta_1}{(w_1\theta_2-w_2\theta_1)}$$

Solution for free-free plates

At $r=r_e$

$$Q_r=0=\alpha(Q_r)_1+\beta(Q_r)_2$$
$$M_r=0=\alpha(M_r)_1+\beta(M_r)_2$$

hence,

$$\alpha=0,\ \beta=0 \text{ and } N=w_3=P/K.$$

3-Solution for the case with a central concentrated force F

The governing differential equation for this case is

$$\left[\nabla^4-\frac{(2-\nu)}{(1-\nu)}\frac{\kappa}{\lambda^2 D}\nabla^2+\frac{\kappa}{D}\right]w = \left[-\frac{(2-\nu)}{(1-\nu)}\frac{\nabla^2}{\lambda^2}-1\right]\frac{F}{D}\delta(x-x_o,y-y_o)$$

i.e

$$(\nabla^4-2b\nabla^2+c)w=\left[-\frac{(2-\nu)}{(1-\nu)}\frac{\nabla^2}{\lambda^2}-1\right]\frac{F}{D}\delta(x-x_o,y-y_o)$$

where

$$b=\frac{(2-\nu)}{2(1-\nu)}\frac{\kappa}{\lambda^2 D},\qquad c=\frac{\kappa}{D}$$

in which (x_o,y_o) is the point of load application. The solution is

$$w=w_{pi}+w_{cf}$$

and should be a non-singular solution to

$$(\nabla^4-2b\nabla^2+c)w=0$$

i.e

$$w_{cf}=\alpha I_o(\lambda_1 r)+\beta I_o(\lambda_2 r)$$

where

$$\begin{pmatrix}\lambda_1^2\\ \lambda_2^2\end{pmatrix}=b\pm\sqrt{b^2-c}$$

the particular integral is a singular solution to the governing differential equation. Let

$$w_{pi}\equiv w(s)$$

using 2-D Fourier transform, where

$$\overline{f}(\xi,\eta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{i[\xi(x-x_o)+\eta(y-y_o)]}dx\,dy$$

$$\rho=\sqrt{\xi^2+\eta^2}$$

the differential equation can be reduced to

$$(\rho^4-2b\rho^2+c)\overline{w}(s)=\left[\frac{(2-\nu)\rho^2}{(1-\nu)\lambda^2}+1\right]\frac{F}{D(2\pi)}$$

i.e

$$\overline{w}^{(s)}=-\frac{F}{2\pi D(\lambda_1^2-\lambda_2^2)}\left[-\frac{\left[\frac{(2-\nu)\lambda_1^2}{(1-\nu)\lambda^2}+1\right]}{\rho^2+\lambda_1^2}-\frac{\left[-\frac{(2-\nu)\lambda_2^2}{(1-\nu)\lambda^2}+1\right]}{\rho^2+\lambda_2^2}\right]$$

i.e

$$\overline{w}^{(s)}=-\left[\frac{\phi_1}{\rho^2+\lambda_1^2}-\frac{\phi_2}{\rho^2+\lambda_2^2}\right]$$

where

$$\phi_1 = -\frac{F}{4\pi D\sqrt{b^2-c}}\left[-\frac{(2-\nu)}{(1-\nu)}\frac{\lambda_1^2}{\lambda^2}+1\right]$$

$$\phi_2 = -\frac{F}{4\pi D\sqrt{b^2-c}}\left[-\frac{(2-\nu)}{(1-\nu)}\frac{\lambda_2^2}{\lambda^2}+1\right]$$

Hence, it can be proved that:

$$w^{(s)} = -\left[\phi_1 w_1^{(s)} - \phi_2 w_2^{(s)}\right]$$

where

$$w_1^{(s)} = K_o(\lambda_1 r)$$

$$w_2^{(s)} = K_o(\lambda_2 r)$$

<u>The effect of $w_i^{(s)} = K_o(\lambda_i r)$</u>

$$K_o'(z) = -K_1(z)$$
$$\nabla^2 K_o(\lambda r) = \lambda^2 K_o(\lambda r)$$

$$K_1'(z) - \left[K_o(z) + \frac{1}{2}K_1(z)\right]$$

$$\frac{dw_i^{(s)}}{dr} = -\lambda_i K_1(\lambda_i r)$$

$$\left(Q_r^{(s)}\right)_i = D\lambda_i A_i K_1(\lambda_i r)$$

where

$$A_i = \lambda_i^2 - \frac{(2-\nu)}{(1-\nu)}\frac{\kappa}{\lambda^2 D}$$

$$(\theta_r^{(s)})_i=\lambda_i\left[\frac{2}{(1-\nu)\lambda^2}A_i+1\right]K_1(\lambda_i r)$$

$$(M_r^{(s)})_i=-D\left[B_i+2\left(\frac{\lambda_i^2}{\lambda^2}\right)A_i\right]K_o(\lambda_i r)-\frac{D\lambda_i}{r}\left[\frac{2A_i}{\lambda^2}+(1-\nu)\right]k_1(\lambda_i r)$$

Hence

$$w^{(s)}=-\left[\phi_1 K_o(\lambda_1 r)-\phi_2 K_o(\lambda_2 r)\right]$$

$$Q_r^s=-\left[\phi_1(Q_r^{(s)})_1-\phi_2(Q_r^{(s)})_2\right]$$

$$\theta_r^s=-\left[\phi_1(\theta_r^{(s)})_1-\phi_2(\theta_r^{(s)})_2\right]$$

$$M_r^s=-\left[\phi_1(M_r^{(s)})_1-\phi_2(M_r^{(s)})_2\right]$$

Solution for a simply supported plate

At $r=r_e$;

$$w=0=\alpha w_1(r_e)+\beta w_2(r_e)+w^s$$
$$M_r=0=\alpha M_1(r_e)+\beta M_2(r_e)+M^s(r_e)$$

$$\alpha=-\left(\frac{w_s M_2-M^s w_2}{w_1 M_2-w_2 M_1}\right)$$

$$\beta=-\left(\frac{w_1 M_s-w_s M_1}{w_1 M_2-w_2 M_1}\right)$$

Clamped plates

$$\alpha=-\frac{w_s\theta_2-\theta^s w_2}{w_1\theta_2-w_2\theta_1}$$

$$\beta=-\frac{w_1\theta_s-w_s\theta_1}{w_1\theta_2-w_2\theta_1}$$

Solution for free-free plates

$$\alpha = -\frac{w_s Q_2 - Q_s w_2}{w_1 Q_2 - w_2 Q_1}$$

$$\beta = -\frac{w_1 Q_s - w_s Q_1}{w_1 Q_2 - w_2 Q_1}$$

G-3 ANALYTICAL SOLUTION FOR THICK SIMPLY SUPPORTED RECTANGULAR PLATES ON ELASTIC FOUNDATIONS

1-Governing equations

This can be deduced from previous work of thick plates without foundation, by replacing "q" with P−Kw, and are summarised as follows:

(i) Displacement

$$D\left[\nabla^4-\frac{(2-\nu)}{(1-\nu)}\frac{\kappa^4}{\lambda^2}\nabla^2+\kappa^4\right]w=-\left[\frac{(2-\nu)}{(1-\nu)}\frac{\nabla^2}{\lambda^2}-1\right]P \tag{1}$$

where

$$\lambda^2=\frac{10}{h^2},\qquad \kappa^4=\frac{K}{D}$$

(ii) Shear forces per unit length

$$(\nabla^2-\lambda^2)Q_x=\lambda^2 D\frac{\partial}{\partial x}\left[\nabla^2-\frac{\kappa^4}{(1-\nu)\lambda^2}\right]w+\frac{1}{(1-\nu)}\frac{\partial P}{\partial x} \tag{2a}$$

$$(\nabla^2-\lambda^2)Q_y=\lambda^2 D\frac{\partial}{\partial y}\left[\nabla^2-\frac{\kappa^4}{(1-\nu)\lambda^2}\right]w+\frac{1}{(1-\nu)}\frac{\partial P}{\partial y} \tag{2b}$$

(iii) Slope angles

$$\theta_x=\frac{2Q_x}{D\lambda^2(1-\nu)}-\frac{\partial w}{\partial x} \tag{3a}$$

$$\theta_y=\frac{2Q_y}{D\lambda^2(1-\nu)}-\frac{\partial w}{\partial y} \tag{3b}$$

using

$$\theta_x=\frac{D(1-\nu)}{2}\lambda^2(\theta_x+\frac{\partial w}{\partial x}) \tag{4a}$$

$$\theta_y=\frac{D(1-\nu)}{2}\lambda^2(\theta_y+\frac{\partial w}{\partial y}) \tag{4b}$$

then, it can be deduced from (2) and (4)

$$(\nabla^2-\lambda^2)\theta_x=\frac{\partial}{\partial x}\left[\frac{(1+\nu)}{(1-\nu)}\nabla^2-\frac{2\kappa^4}{(1-\nu)^2\lambda^2}+\lambda^2\right]w+\frac{2}{(1-\nu)^2\lambda^2 D}\frac{\partial P}{\partial x} \tag{5a}$$

$$(\nabla^2-\lambda^2)\theta_y=\frac{\partial}{\partial y}\left[\frac{(1+\nu)}{(1-\nu)}\nabla^2-\frac{2\kappa^4}{(1-\nu)^2\lambda^2}+\lambda^2\right]w+\frac{2}{(1-\nu)^2\lambda^2 D}\frac{\partial P}{\partial y} \tag{5b}$$

(iv) Moments

(a) In terms of shear forces:

$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+\frac{2}{\lambda^2}\frac{\partial Q_x}{\partial x}-\frac{\nu}{(1-\nu)\lambda^2}(P-Kw) \tag{6a}$$

$$M_y=-D\left(\nu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\frac{2}{\lambda^2}\frac{\partial Q_y}{\partial y}-\frac{\nu}{(1-\nu)\lambda^2}(P-Kw) \tag{6b}$$

$$M_{xy}=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}+\frac{1}{\lambda^2}\left(\frac{\partial Q_x}{\partial y}+\frac{\partial Q_y}{\partial x}\right) \tag{6c}$$

(b) In terms of slopes:

$$M_x=D\left(\frac{\partial\theta_x}{\partial x}+\nu\frac{\partial\theta_y}{\partial y}\right)+\frac{\nu}{(1-\nu)\lambda^2}(P-Kw) \tag{7a}$$

$$M_y=D\left(\nu\frac{\partial\theta_x}{\partial x}+\frac{\partial\theta_y}{\partial y}\right)+\frac{\nu}{(1-\nu)\lambda^2}(P-Kw) \tag{7b}$$

$$M_{xy}=\frac{D(1-\nu)}{2}\left(\frac{\partial\theta_x}{\partial y}+\frac{\partial\theta_y}{\partial x}\right) \tag{7c}$$

2– Simply supported plate under uniform distributed loading

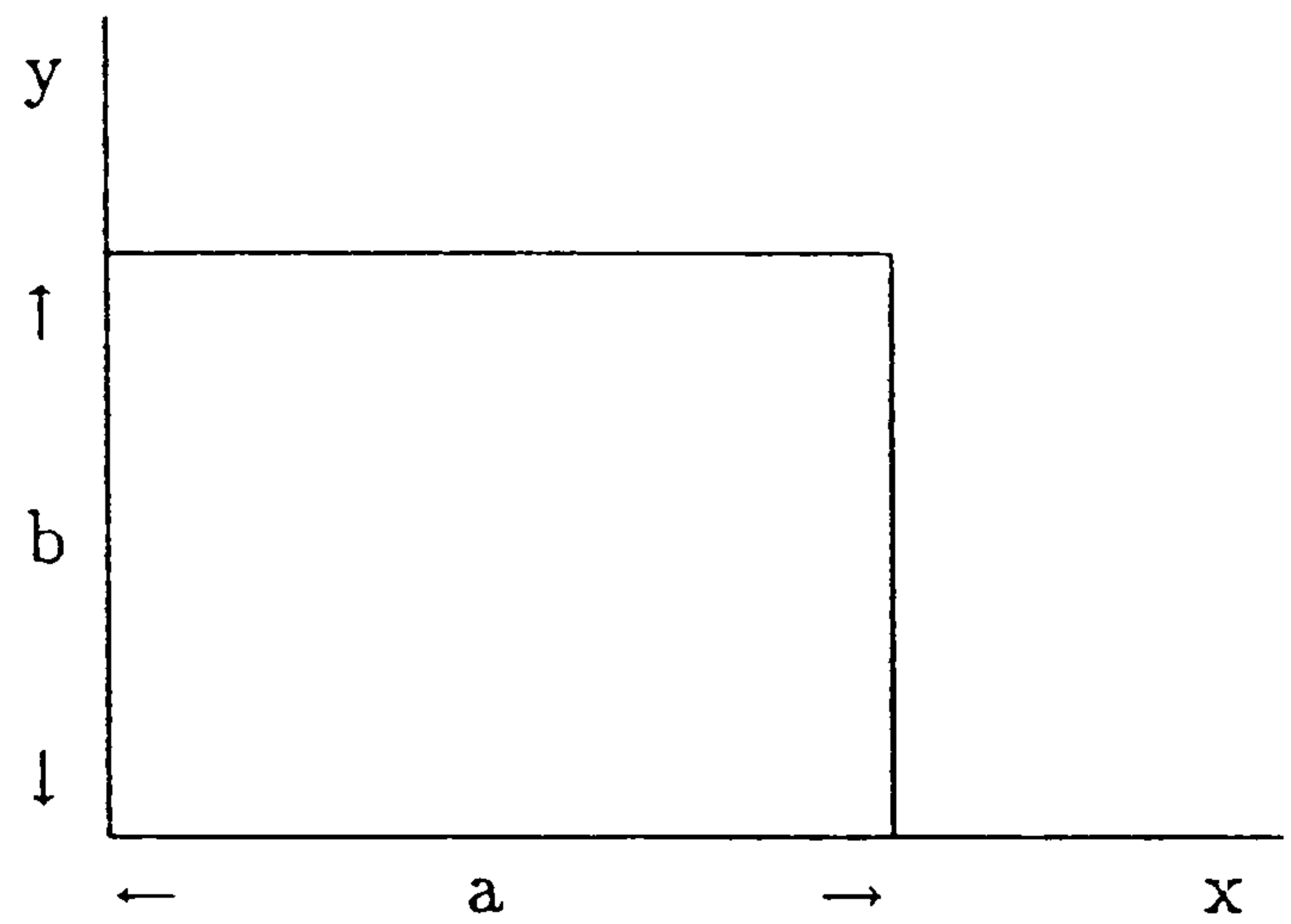
Using a double Fourier series:

$$P=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\alpha_m x\,\sin\beta_n y \tag{8}$$

for the rectangular plate shown in figure below where

$\alpha_m = m\frac{\pi}{a}$

$\beta_n = n\frac{\pi}{b}$



then the displacement will be

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn} \sin\alpha_m x \sin\beta_n y \tag{9}$$

and simply supported boundary conditions are satisfied

<u>To find a_{mn}</u>

Multiply (8) by $\sin\alpha_m x$ $\sin\beta_n y$ and integrate, using orthogonal properties of sine functions, it can be deduced that:

$$\int_0^b\int_0^a p \sin\alpha_m x \sin\beta_n y \, dx \, dy = a_{mn}\int_0^b\int_0^a \sin^2\alpha_m x \sin^2\beta_n y \, dx \, dy$$

let

$\theta = \frac{\pi}{a}x$

$\phi = \frac{\pi}{b}y$

then

$$\int_0^b\int_0^a p \sin\alpha_m x \sin\beta_n y \, dx \, dy = \left(\frac{ab}{\pi^2}\right)p\int_0^{\pi}\int_0^{\pi} p \sin_m\theta \sin_n\phi \, d\theta \, d\phi$$

$$=\left(\frac{ab}{\pi^2}\right)\frac{p}{mn}\left[1-\cos(m\pi)\right]\cdot\left[1-\cos(n\pi)\right]$$

$$\left.\begin{aligned} &=0 && \text{for even } m \text{ or even } n \\ &=\left(\frac{ab}{\pi^2}\right)\frac{4p}{mn} && \text{for odd } m \text{ and odd } n \end{aligned}\right\}(a)$$

$$a_{mn}\int_0^b\int_0^a p\,\sin^2(\alpha_m x)\sin^2(\beta_n y)\,dx\,dy=\left(\frac{ab}{\pi^2}\right)a_{mn}\int_0^\pi\int_0^\pi\left[\frac{1-\cos^2(2m\theta)}{2}\right]\cdot\left[\frac{1-\cos^2(2m\phi)}{2}\right]d\phi\, d\theta$$

$$=\left(\frac{ab}{\pi^2}\right)a_{mn}\frac{\pi^2}{4} \tag{b}$$

comparing (a) with (b), then;

$$a_{mn}=0 \qquad \text{for } m \text{ even or } n \text{ even}$$

$$=\frac{16P}{\pi^2(mn)} \qquad \text{for } m \text{ odd and } n \text{ odd}$$

i.e

$$P=\sum_{m=1,3,5}^{\infty}\ \sum_{n=1,3,5}^{\infty} a_{mn}\sin\alpha_m x\,\sin\beta_n y \tag{8$'$}$$

$$w=\sum_{m=1,3,5}^{\infty}\ \sum_{n=1,3,5}^{\infty} w_{mn}\sin\alpha_m x\,\sin\beta_n y \tag{9$'$}$$

<u>Notice that:</u>

$$\nabla^2 \sin\alpha_m x\,\sin\beta_n y=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\sin\alpha_m x\,\sin\beta_n y$$

$$=-(\alpha_m^2+\beta_n^2)\sin\alpha_m x\,\sin\beta_n y \tag{10}$$

Substituting from $(8)'$ and $(9)'$ into (1), and equating the coefficients of $\sin\alpha_m x\,\sin\beta_n y$ in the two sides of the equation, then it can be proved that:

$$D\left[(\alpha_m^2+\beta_n^2)^2+\frac{(2-\nu)}{(1-\nu)}\frac{\kappa^4}{\lambda^2}(\alpha_m^2+\beta_n^2)+\kappa^4\right]w_{mn}=\left[\frac{(2-\nu)}{(1-\nu)\lambda^2}(\alpha_m^2+\beta_n^2)+1\right]a_{mn}$$

i.e

$$w_{mn}=a_{mn}\frac{\alpha}{\beta} \tag{11}$$

where

$$a_{mn}=\frac{16P}{\pi^2(mn)}$$

$$\alpha=\frac{(2-\nu)}{(1-\nu)\lambda^2}(\alpha_m^2+\beta_n^2)+1$$

$$\beta=(\alpha_m^2+\beta_n^2)^2+\frac{(2-\nu)\kappa^4}{(1-\nu)\lambda^2}(\alpha_m^2+\beta_n^2)+\kappa^4$$

3-<u>Simply supported plate under central concentrated loading</u>

For such a case

$$P=F\delta(x-\frac{a}{2},y-\frac{b}{2})$$

$$\equiv\sum_m^\infty\sum_n^\infty a_{mn}\sin(\alpha_m x)\sin(\beta_n x)$$

$$\int_0^b\int_0^a p\ \sin\alpha_m x \sin\beta_n y\ dx\ dy=\int_0^b\int_0^a F\delta(x-\frac{a}{2},y-\frac{b}{2})\ \sin\alpha_m x\sin\beta_n y\ dx\ dy$$

$$=F\ \sin\left(\frac{m\pi}{a}\frac{a}{2}\right)\times\sin\left(\frac{n\pi}{b}\frac{b}{2}\right)$$

$$=F\ \sin\left(\frac{m\pi}{2}\right)\times\sin\left(\frac{n\pi}{2}\right)$$

$$\equiv\left(\frac{ab}{\pi^2}\right)\frac{\pi^2}{4}a_{mn}$$

hence

$$a_{mn}=\frac{4F}{ab}\ \sin\left(\frac{m\pi}{2}\right)\times\sin\left(\frac{n\pi}{2}\right)$$

$$w_{mn}=a_{mn}\frac{\alpha}{\beta}$$

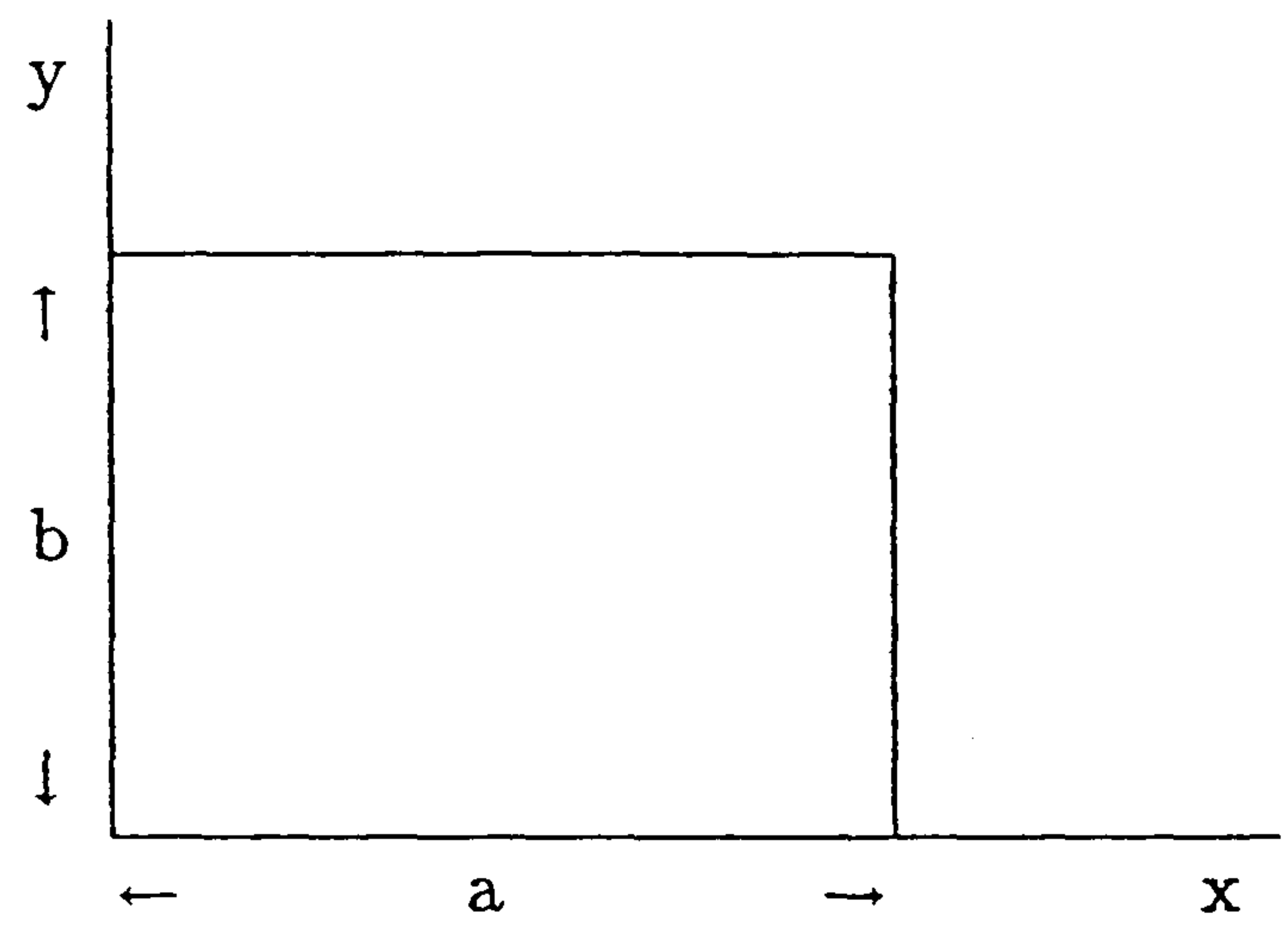
where

α, β are as derived before.

G-4 ANALYTICAL SOLUTION OF THIN SIMPLY-SUPPORTED RECTANGULAR PLATES ON ELASTIC FOUNDATIONS

1. Simply supported plate under uniformly distributed loading

Consider the simply supported plate shown in the figure below,



the governing differential equation is

$$(D\nabla^2+\kappa)w=p \tag{1}$$

using the double Fourier series,

$$p=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\alpha_m x \sin\beta_n y \tag{2}$$

where

$$\alpha_m = m\pi/a$$
$$\beta_n = n\pi/b$$

then the displacement w which satisfies simply supported edge conditions can be expressed as follows:

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn} \sin(\alpha_m x) \sin(\beta_n y) \tag{3}$$

Using an analysis similar to that used for thick plates, it can be proved that:

$$w_{mn}=\frac{\alpha_{mn}}{D(\alpha_m^2+\beta_n^2)^2+\kappa} \tag{4}$$

where

$\alpha_{mn} = 0$ for even m or even n.

$= \frac{16P}{mn\pi^2}$ for odd m or odd n.

2. Simply supported plate under central concentrated loading

For such a case, the concentrated force may be represented in terms of distributed loading as follows:

$$P=F\delta(x-\frac{a}{2},y-\frac{b}{2})$$

$$\equiv\sum_m^\infty\sum_n^\infty a_{mn}\sin(\alpha_m x)\sin(\beta_n x)$$

i.e.

$$\alpha_{mn}=\frac{4F\sin(\frac{m\pi}{2})\sin(\frac{n\pi}{2})}{ab} \tag{6}$$

and

$$w_{mn}=\frac{\alpha_{mn}}{D(\alpha_m^2+\beta_n^2)^2+\kappa}$$

Equation (3) may then be used for the evaluation of w at any (x,y). Note that for both cases:

$$\frac{\partial w}{\partial x}=\sum_m^\infty\ \sum_n^\infty \alpha_m\ w_{mn}\ \cos(\alpha_m x)\ \sin(\beta_n y)$$

and

$$\frac{\partial w}{\partial y}=\sum_m^\infty\ \sum_n^\infty \beta_n\ w_{mn}\ \sin(\alpha_m x)\ \cos(\beta_n y)$$

APPENDIX H

ABES Package
Run on 15-MAR-1991 11:13:45

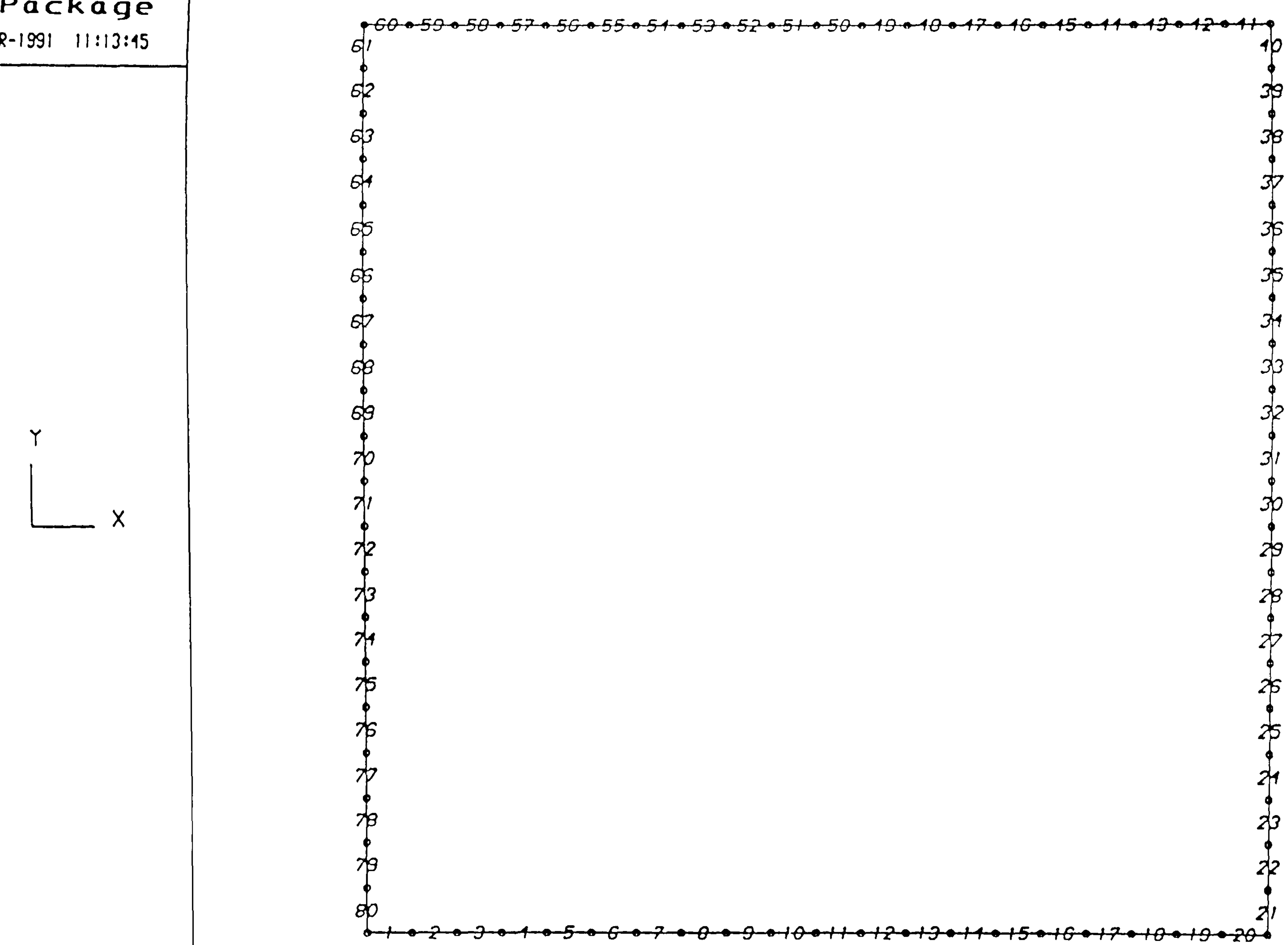


FIG [H.1] BOUNDARY ELEMENT MESH FOR SQUARE PLATE CASE STUDIES

ABES Package
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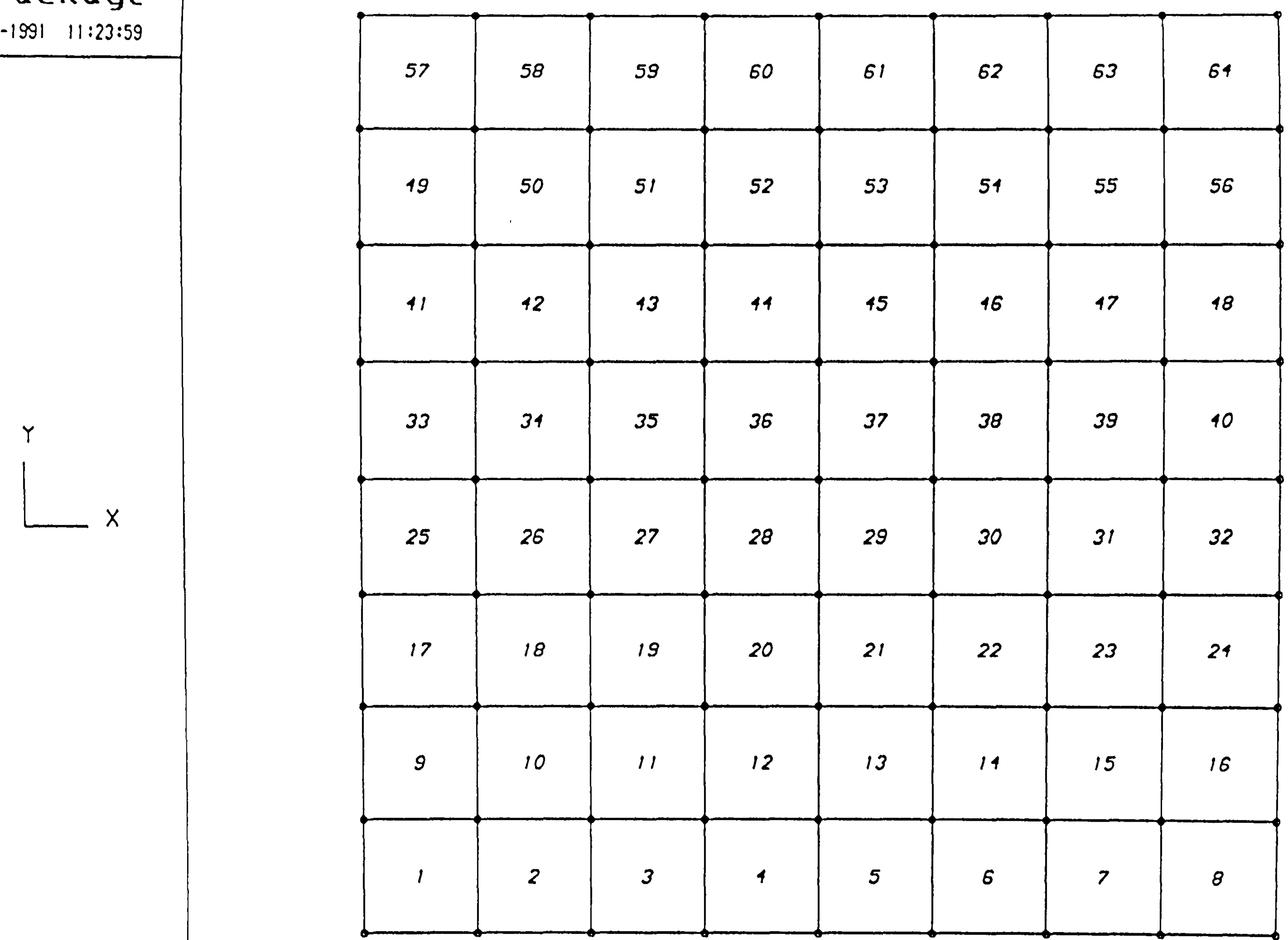


FIG [H.2] FINITE ELEMENT MESH FOR SQUARE PLATE CASE STUDIES

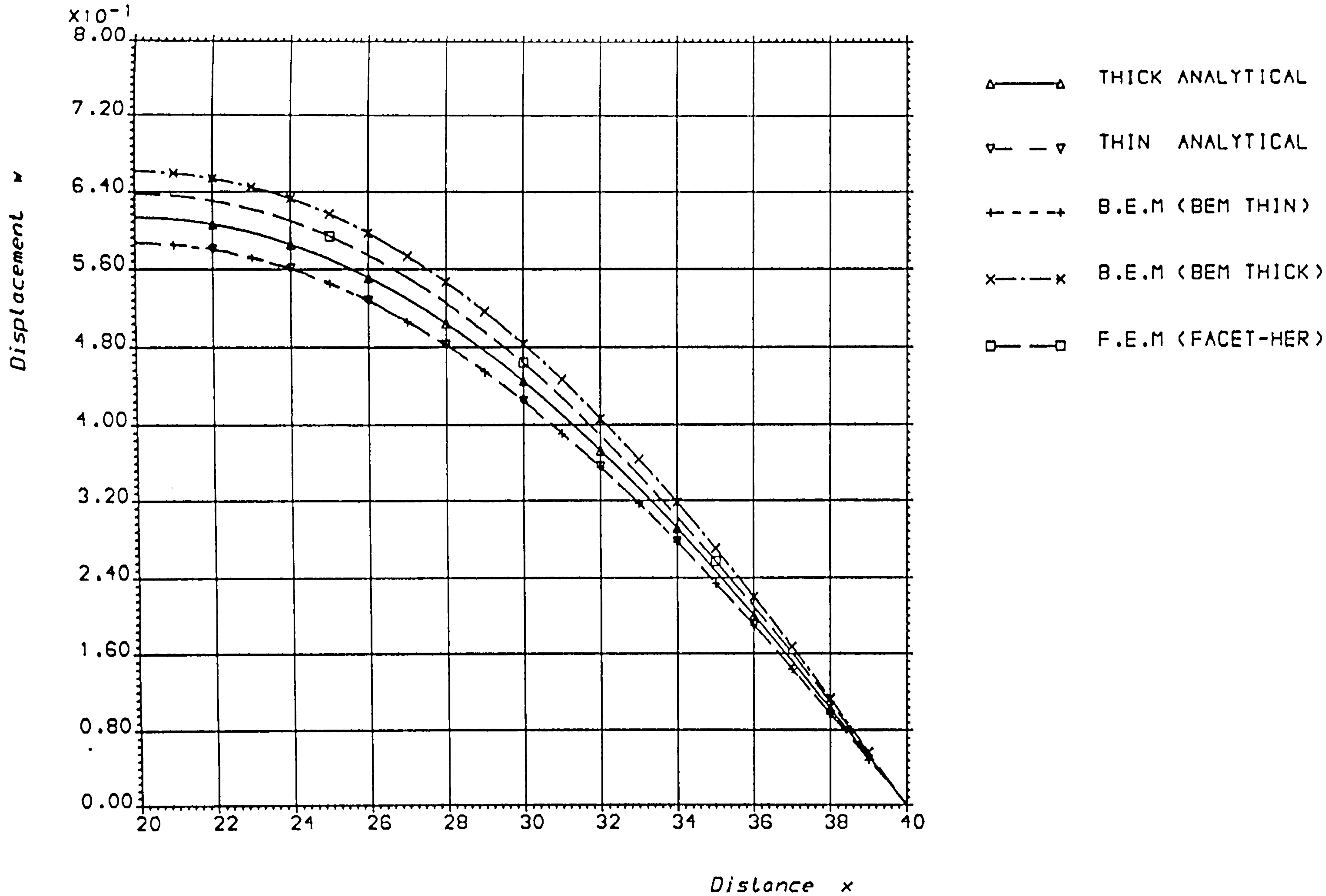


FIG (H.3) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(DOMAIN LOADING CASE WITH FOUNDATION K=200, THICKNESS h=4)

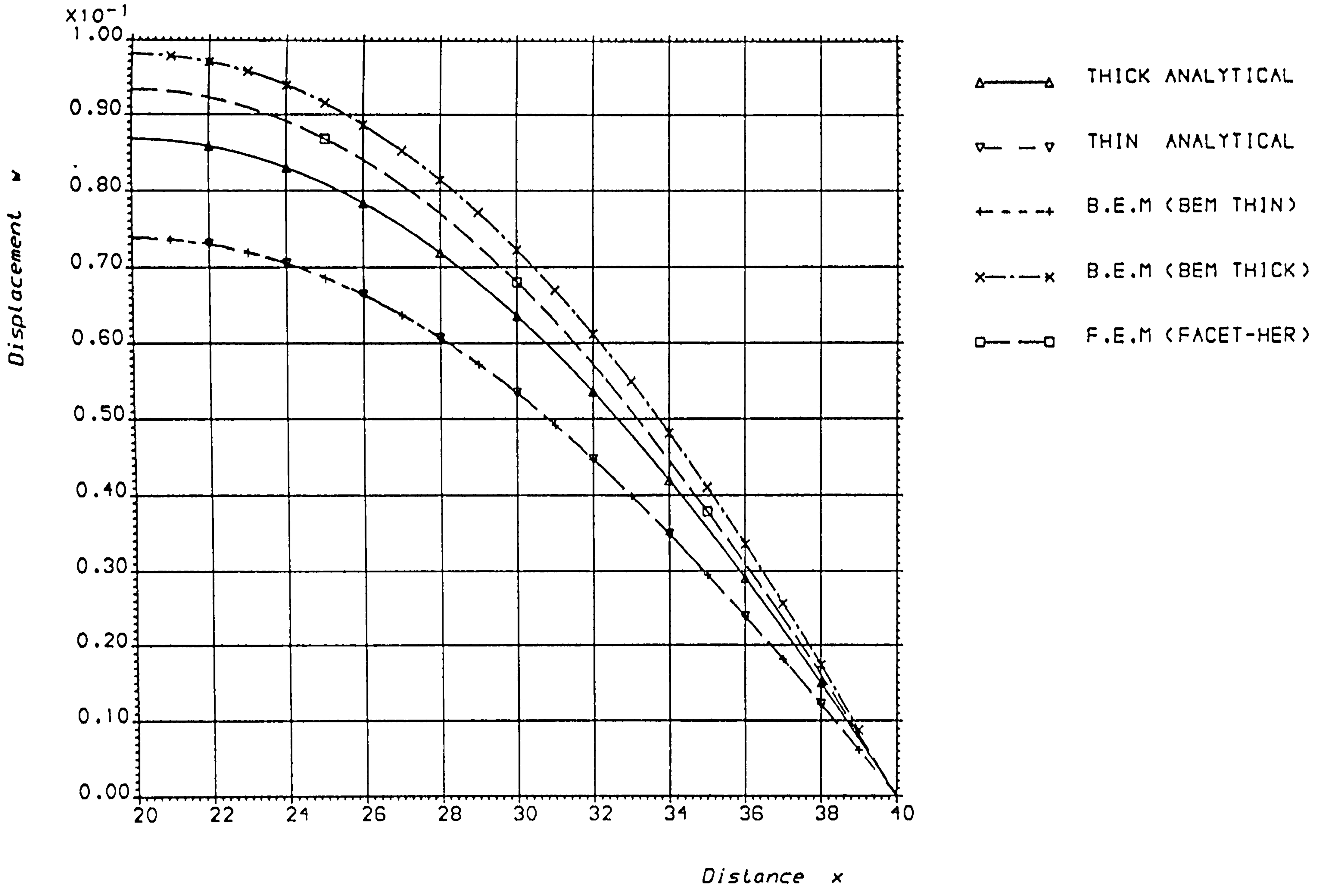


FIG (H.4) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(DOMAIN LOADING CASE WITH FOUNDATION K=200, THICKNESS h=8)

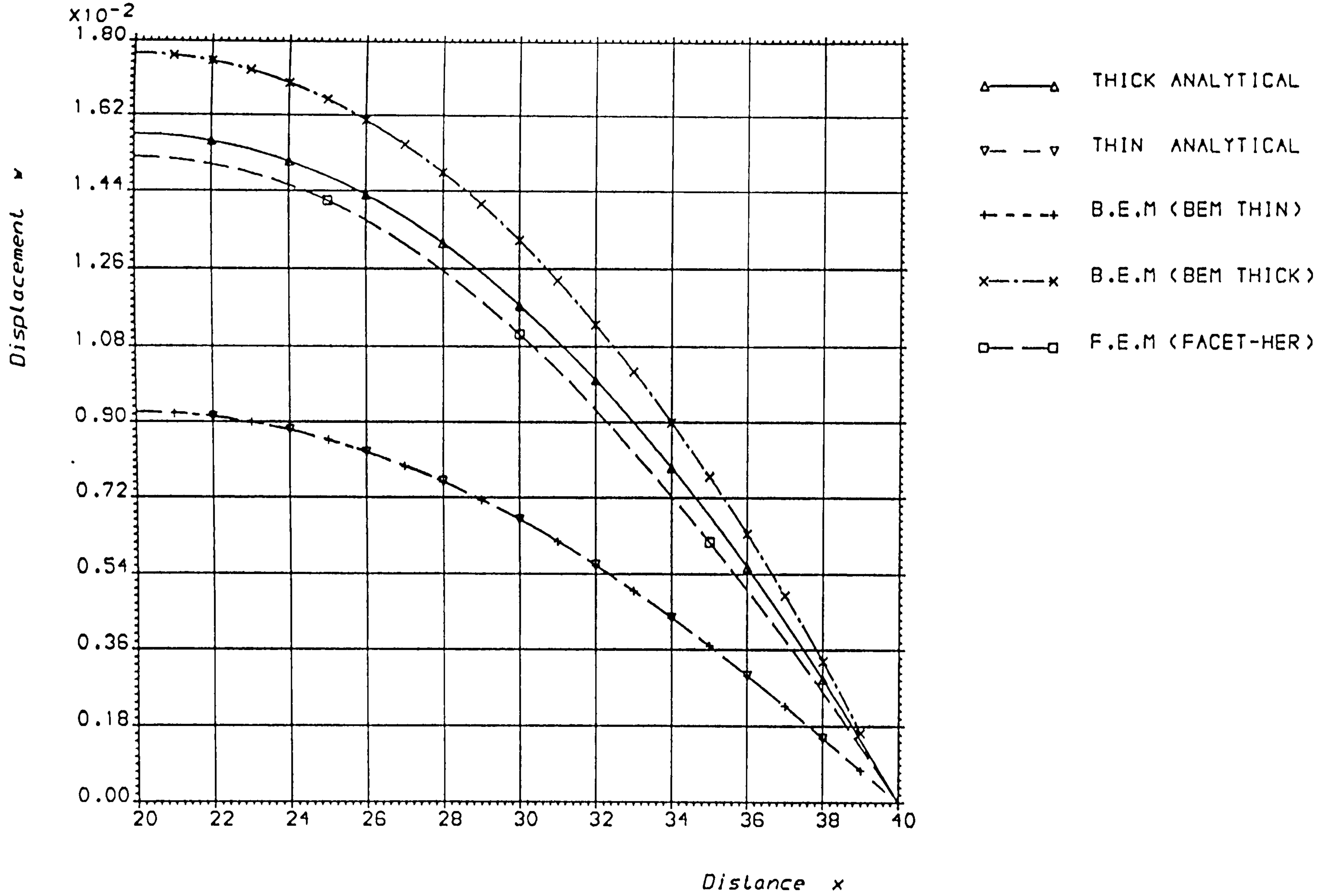


FIG (H.5) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP. RECTANGULR PLATE (DOMAIN LOADING CASE WITH FOUNDATION K=200, THICKNESS h=16)

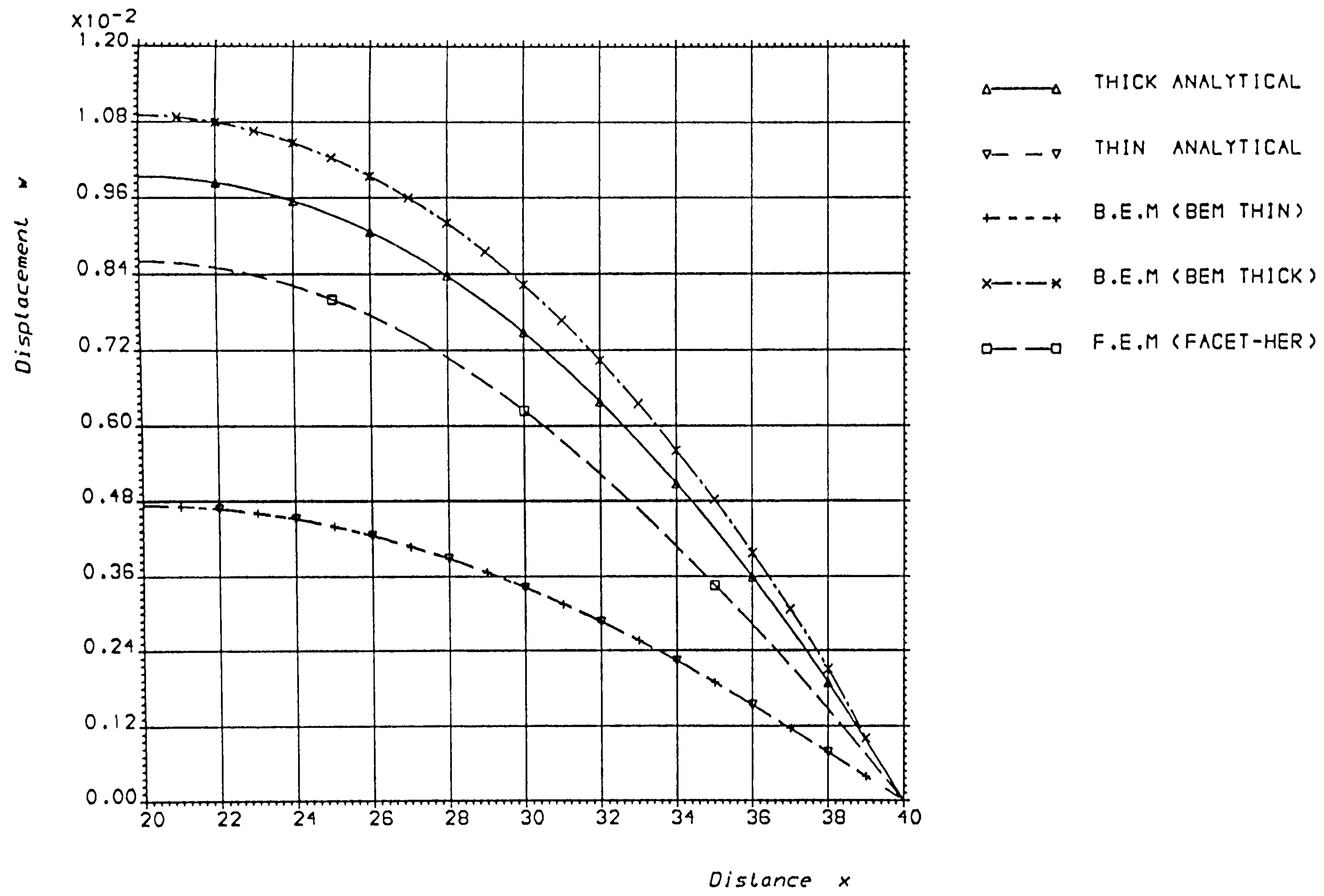


FIG (H.6) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP. RECTANGULR PLATE
(DOMAIN LOADING CASE WITH FOUNDATION K=200, THICKNESS h=20)

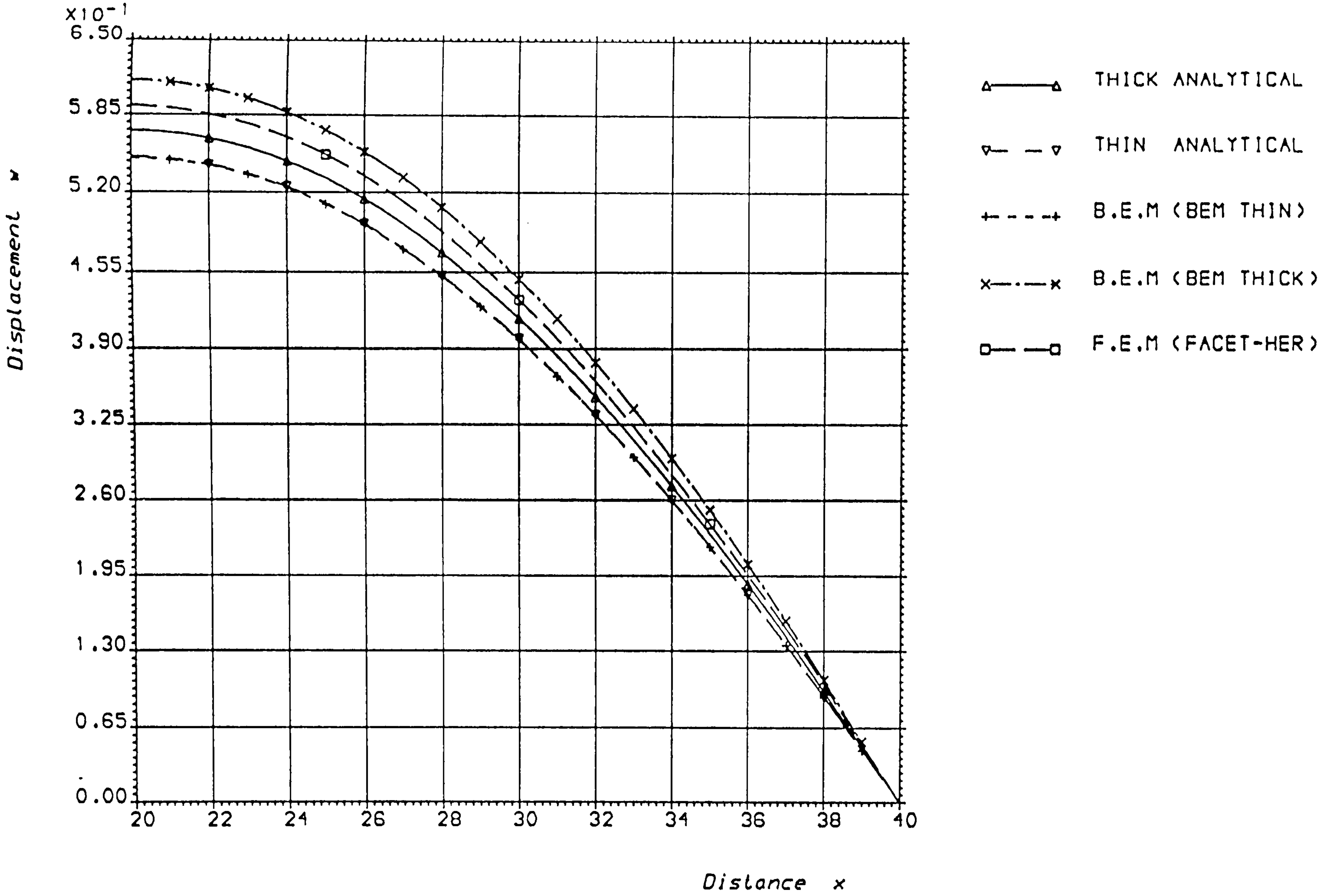


FIG (H.7) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP. RECTANGULR PLATE
(DOMAIN LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=4)

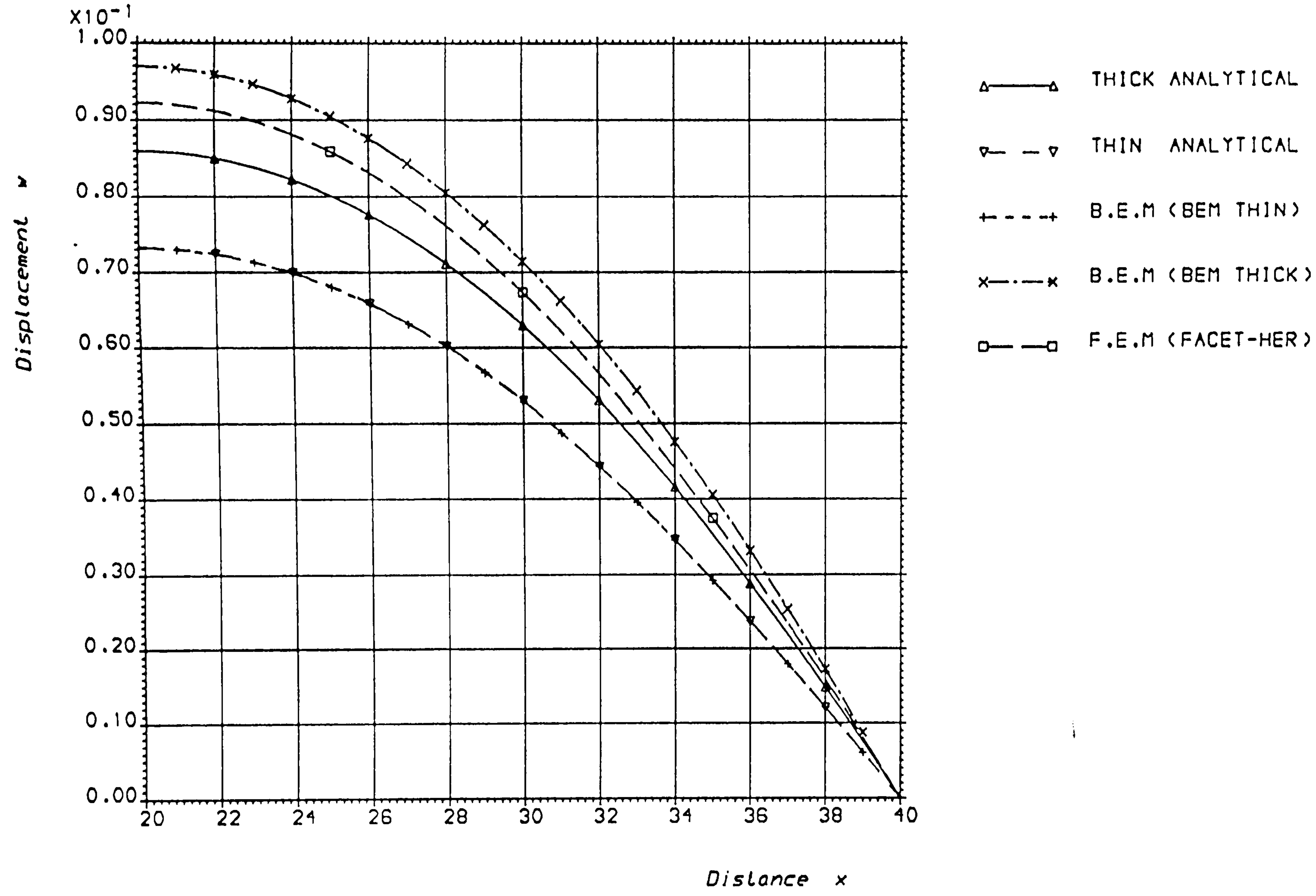


FIG (H.8) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP. RECTANGULR PLATE
(DOMAIN LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=8)

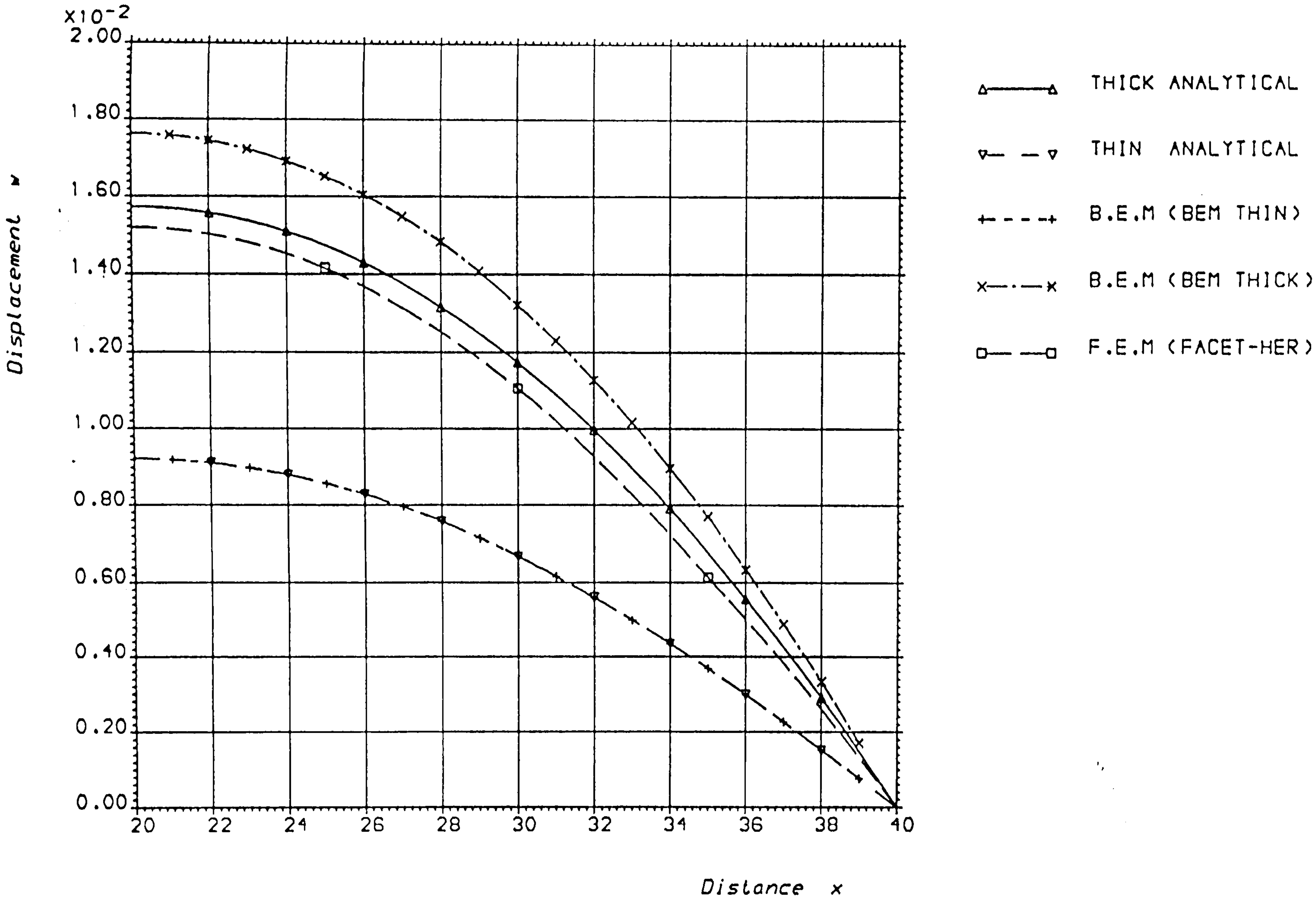


FIG (H.9) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUPP. RECTANGULR PLATE
(DOMAIN LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=16)

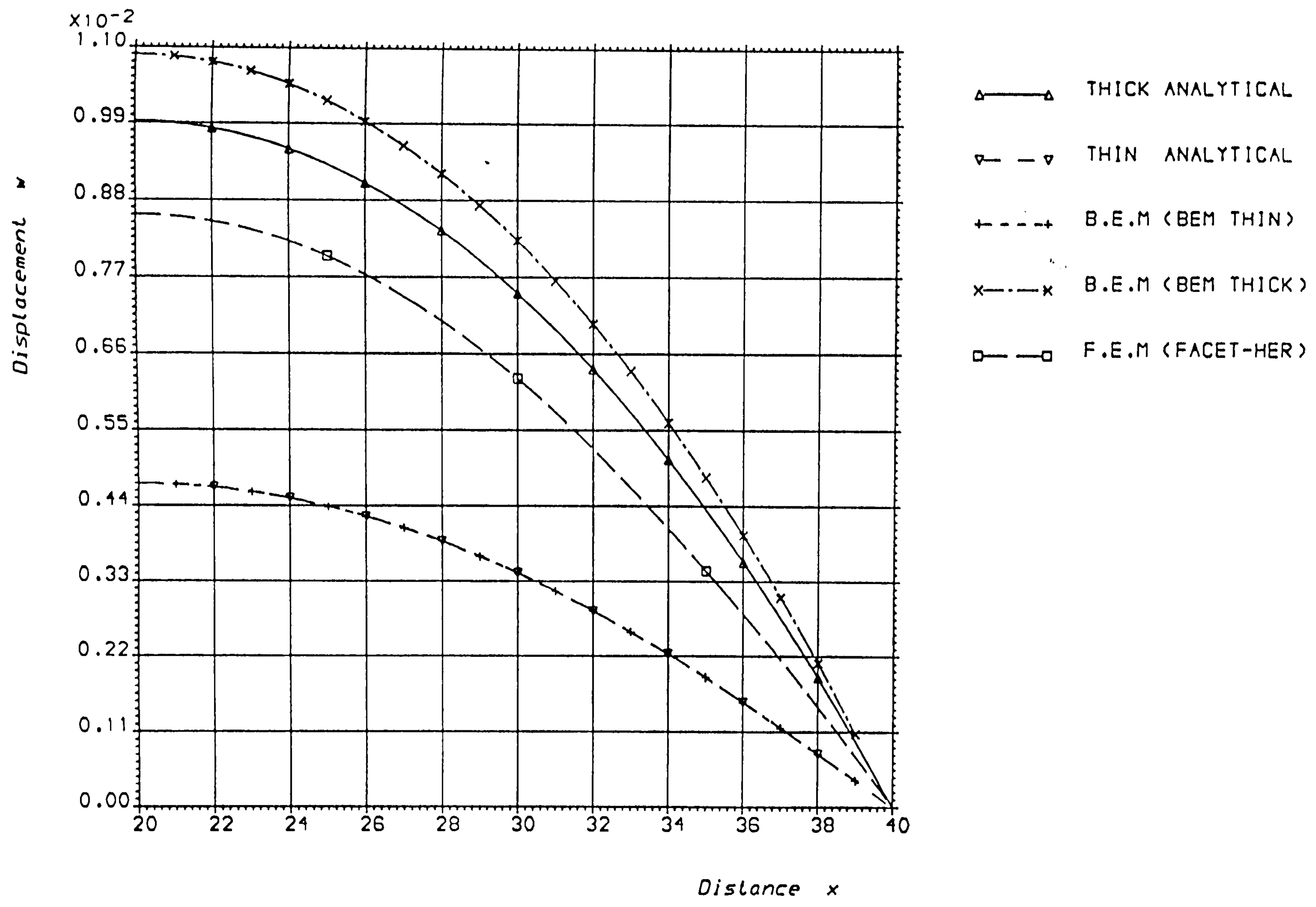


FIG (H.10) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUPP. RECTANGULR PLATE
(DOMAIN LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=20)

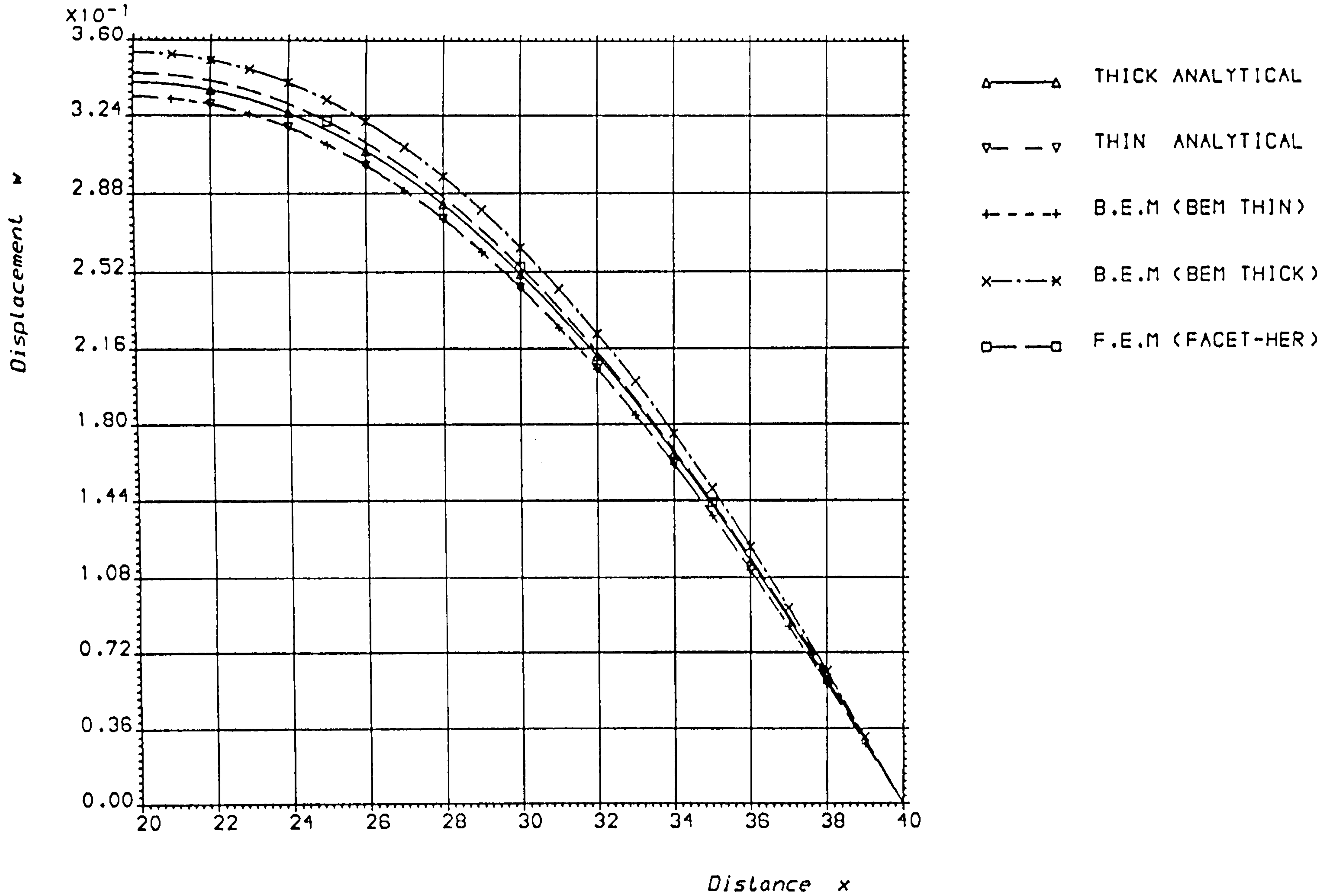
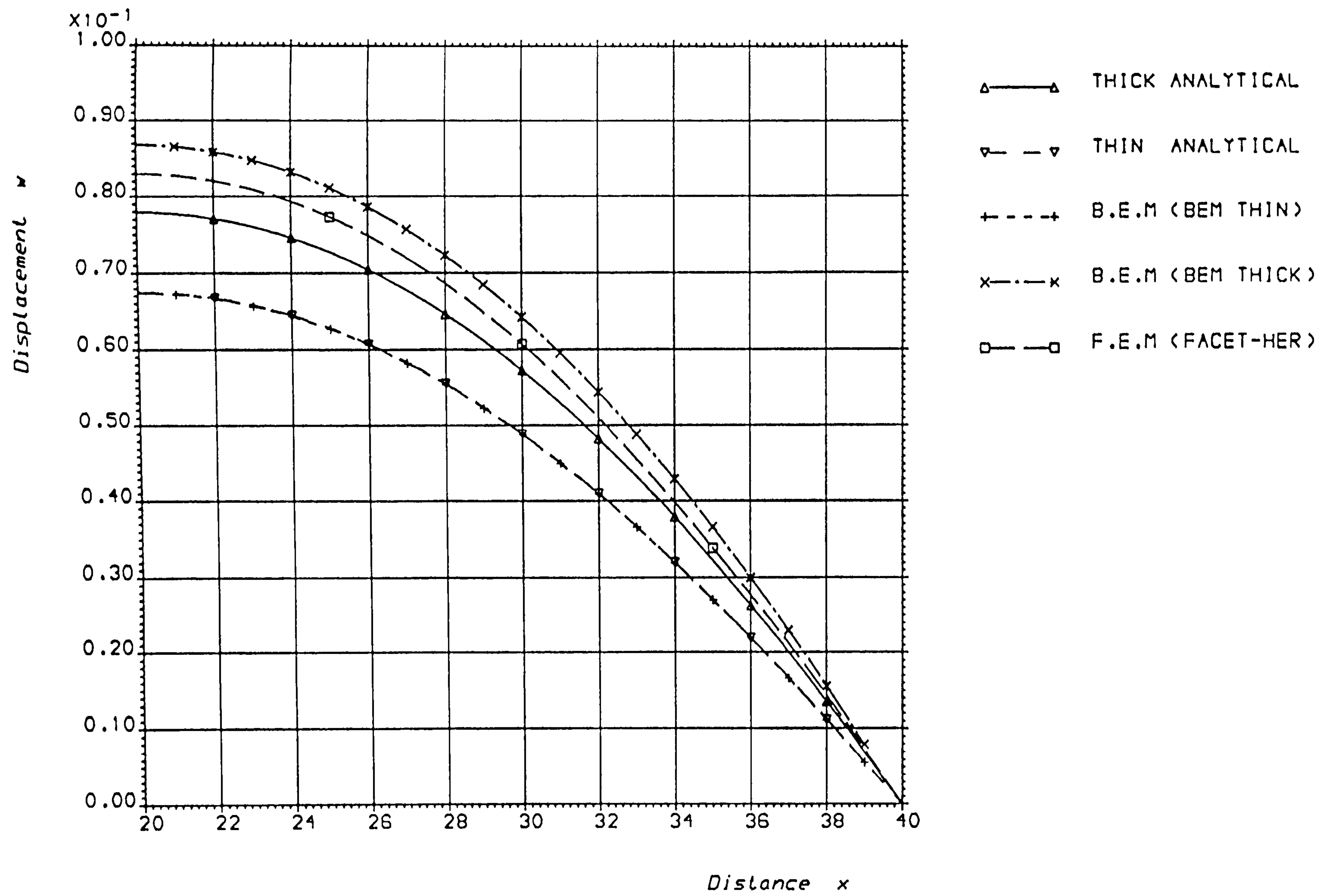


FIG (H.11) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP. RECTANGULR PLATE (DOMAIN LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=4)



..FIG (H.12) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUPP. RECTANGULR PLATE (DOMAIN LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=8)

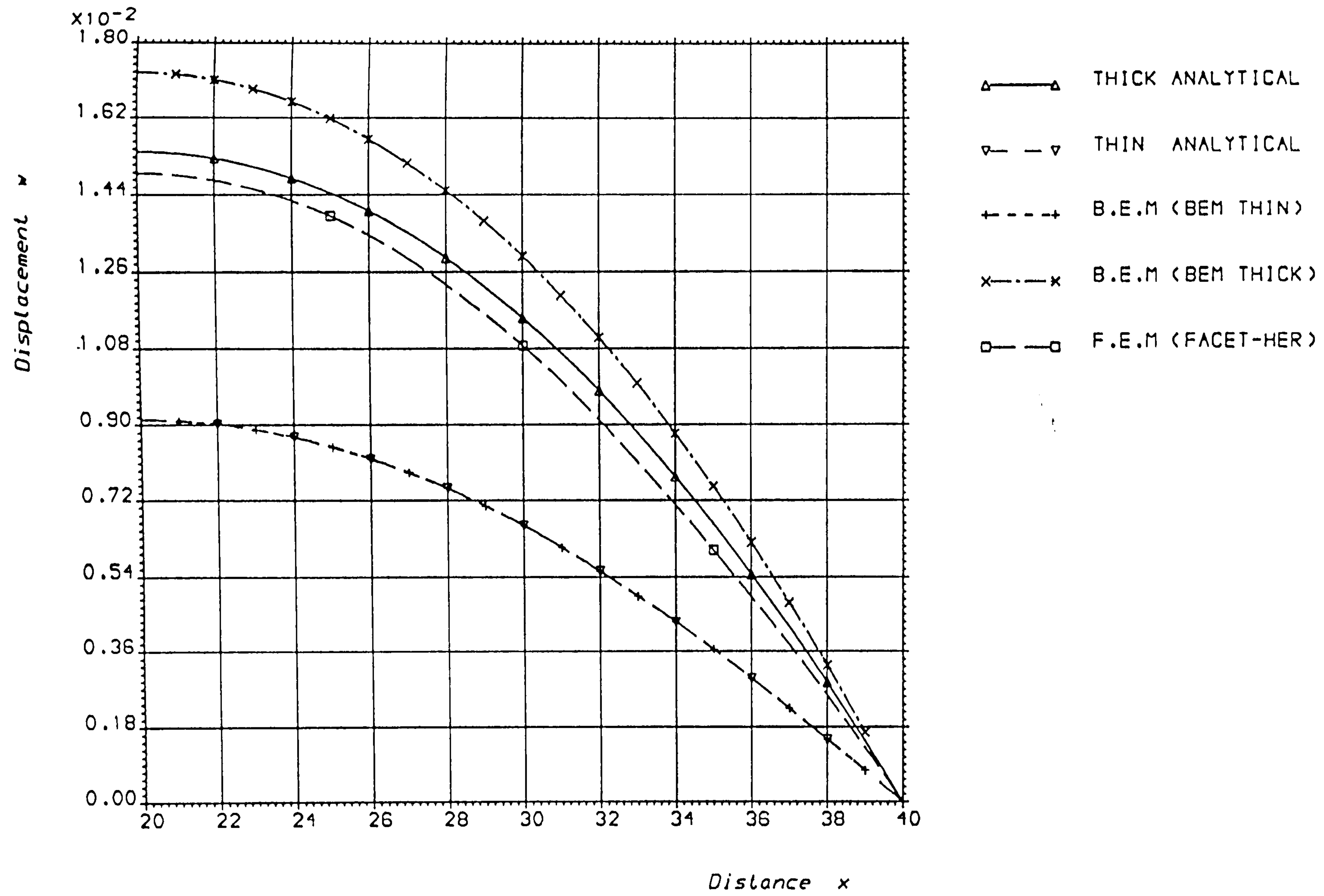


FIG (H.13) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUPP. RECTANGULR PLATE (DOMAIN LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=16)

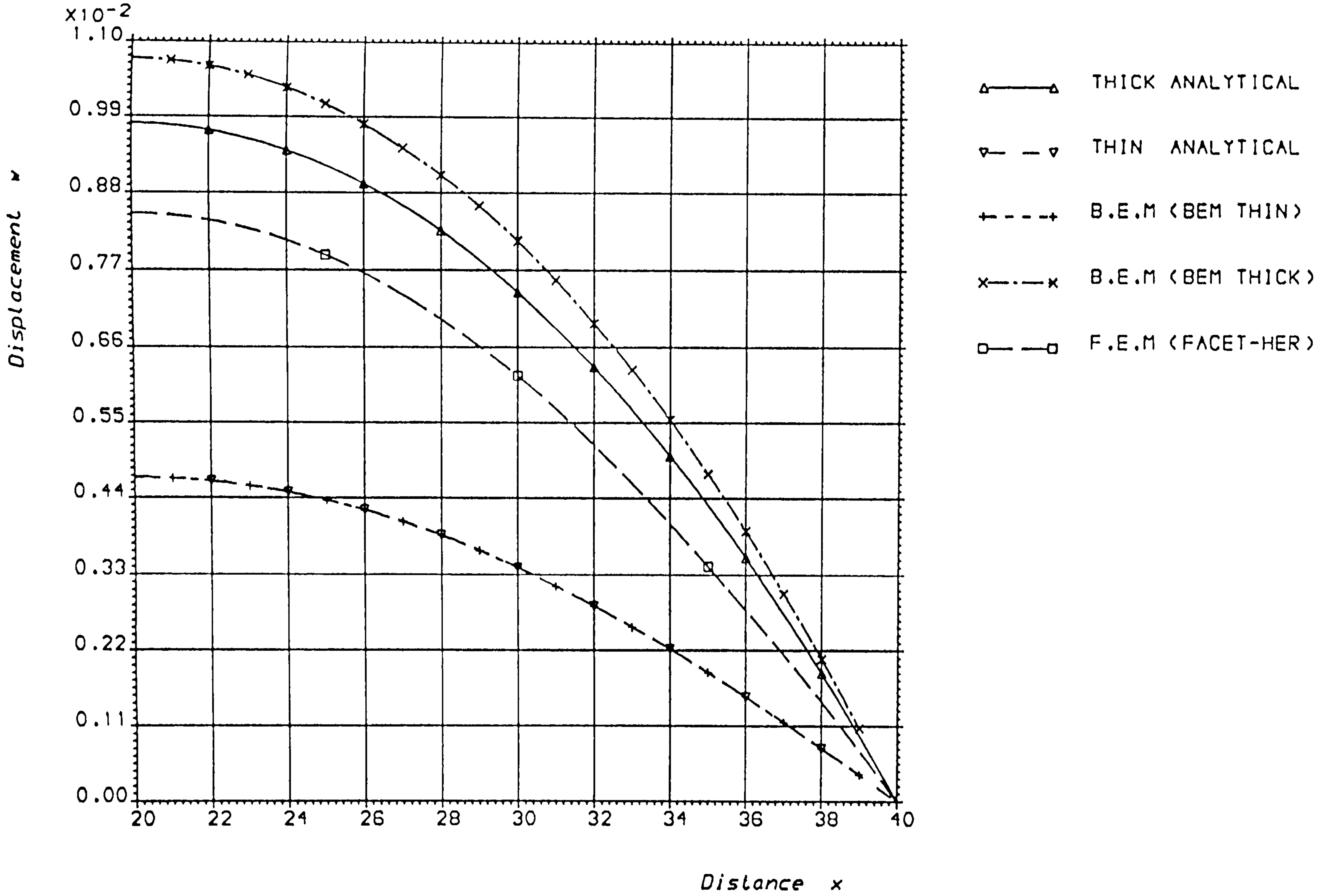


FIG (H.14) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUPP. RECTANGULR PLATE (DOMAIN LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=20)

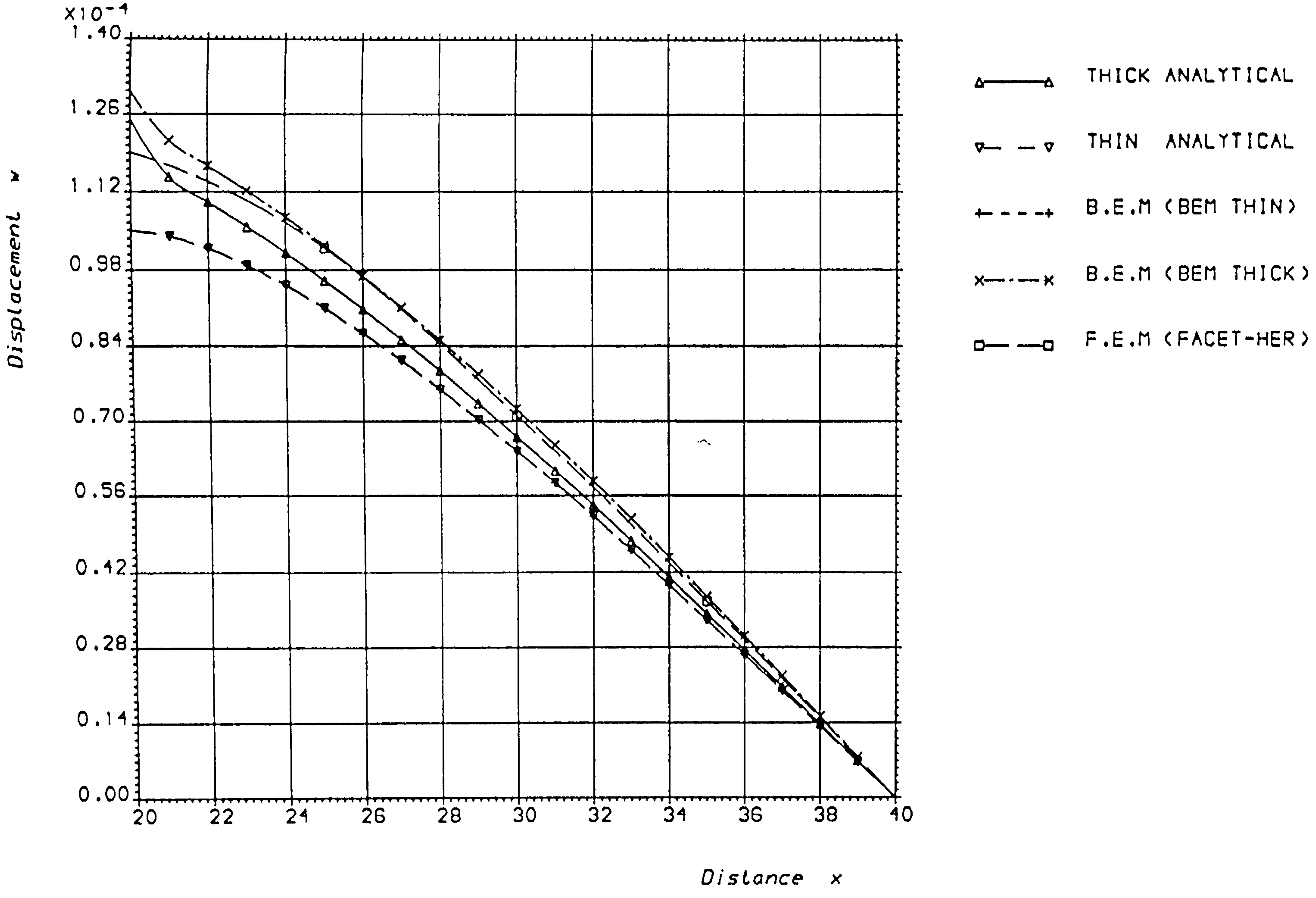


FIG (H.15) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE (CONC. LOADING CASE WITH FOUNDATION K=200, THICKNESS h=4)

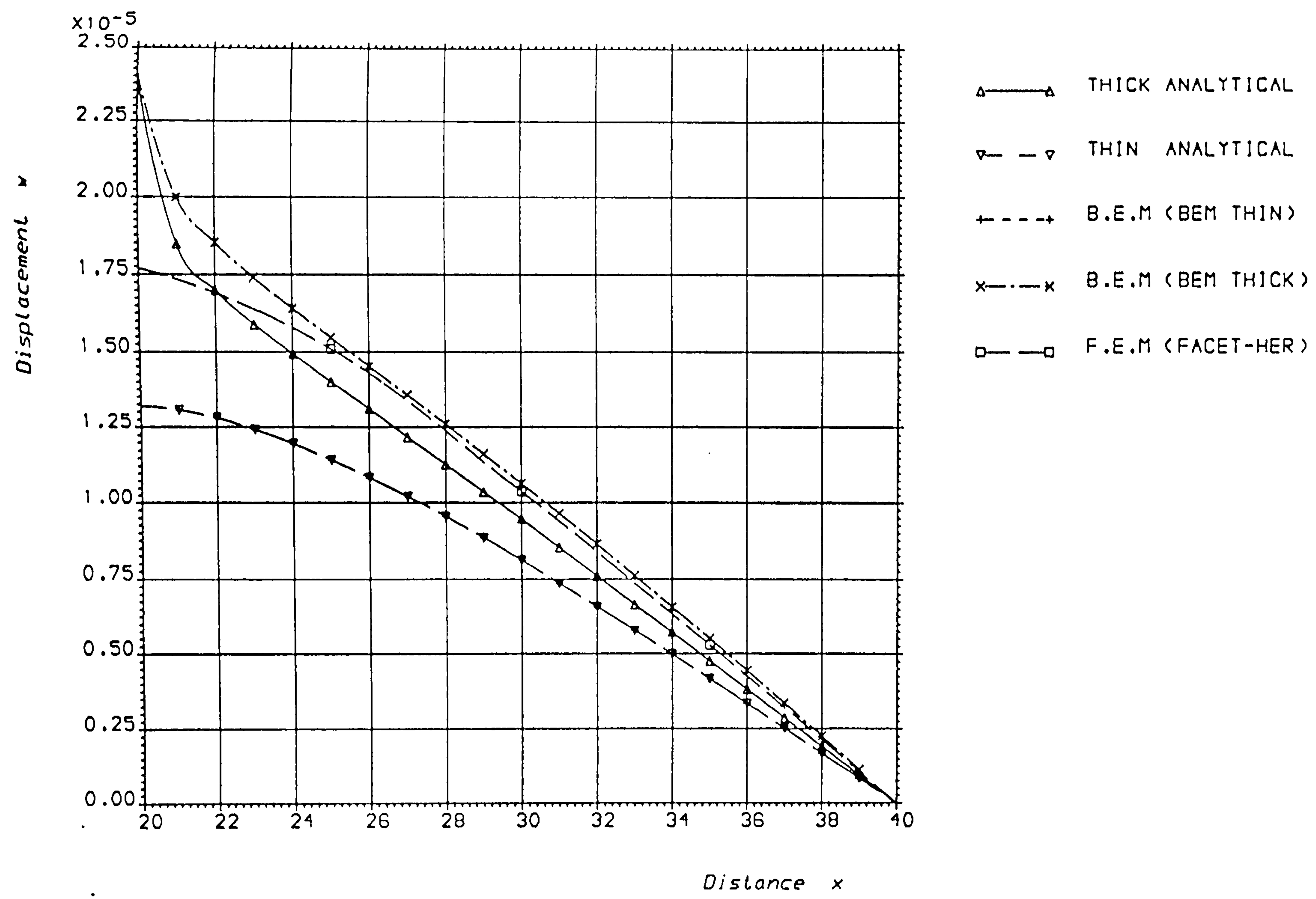


FIG (H.16) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE (CONC. LOADING CASE WITH FOUNDATION K=200, THICKNESS h=8)

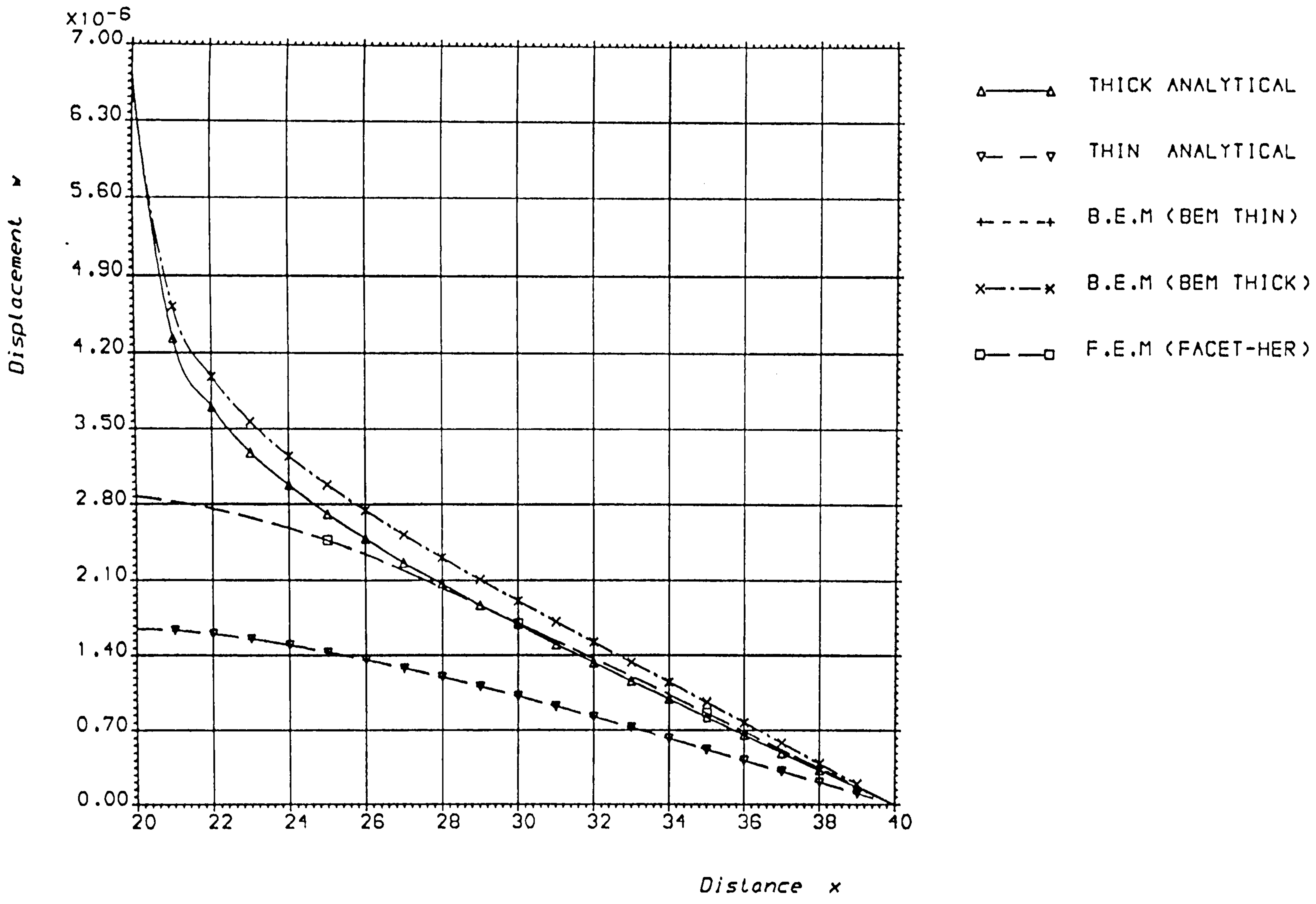


FIG (H.17) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(CONC. LOADING CASE WITH FOUNDATION K=200, THICKNESS K=16)

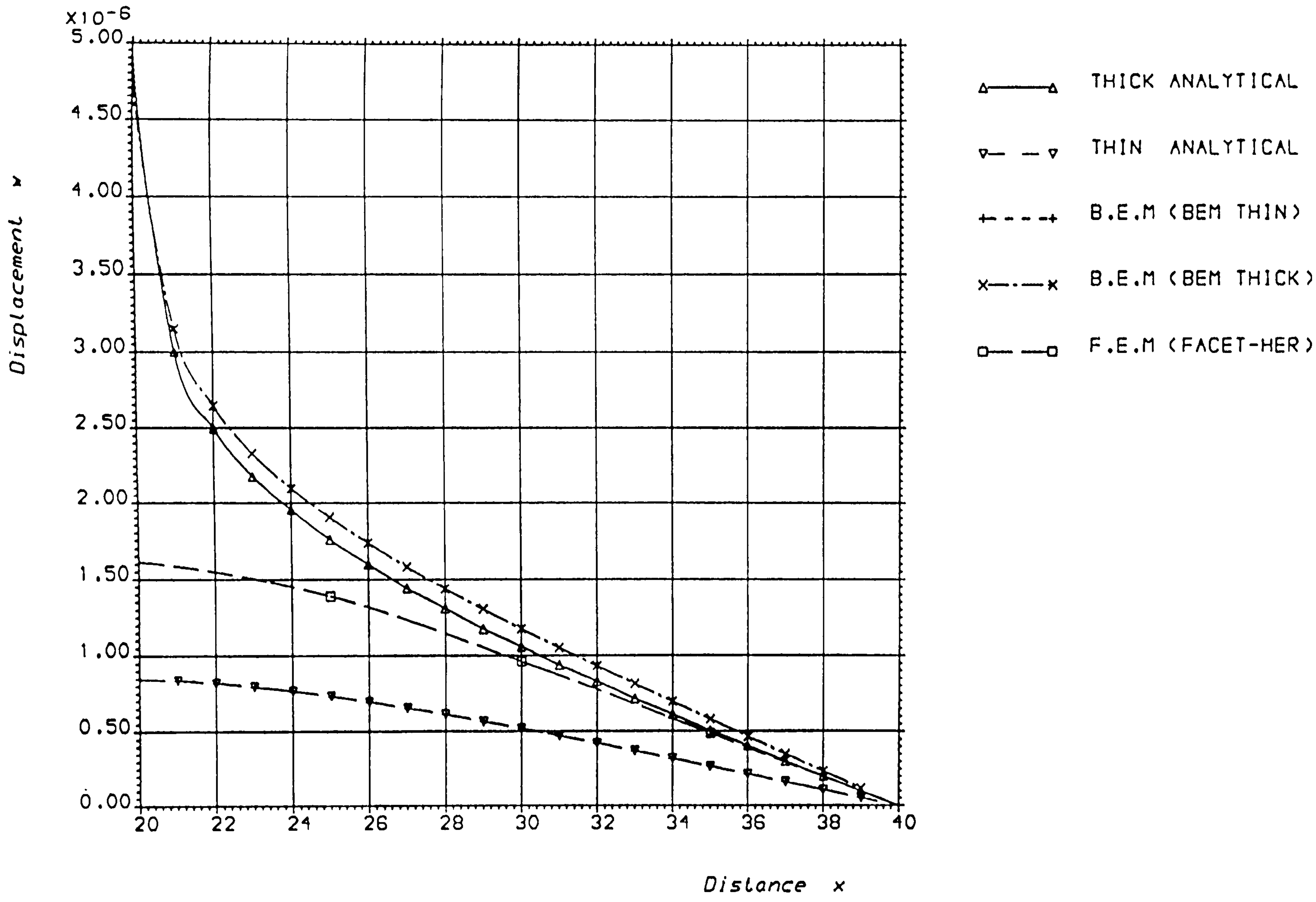


FIG (H.18) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(CONC. LOADING CASE WITH FOUNDATION K=200, THICKNESS h=20)

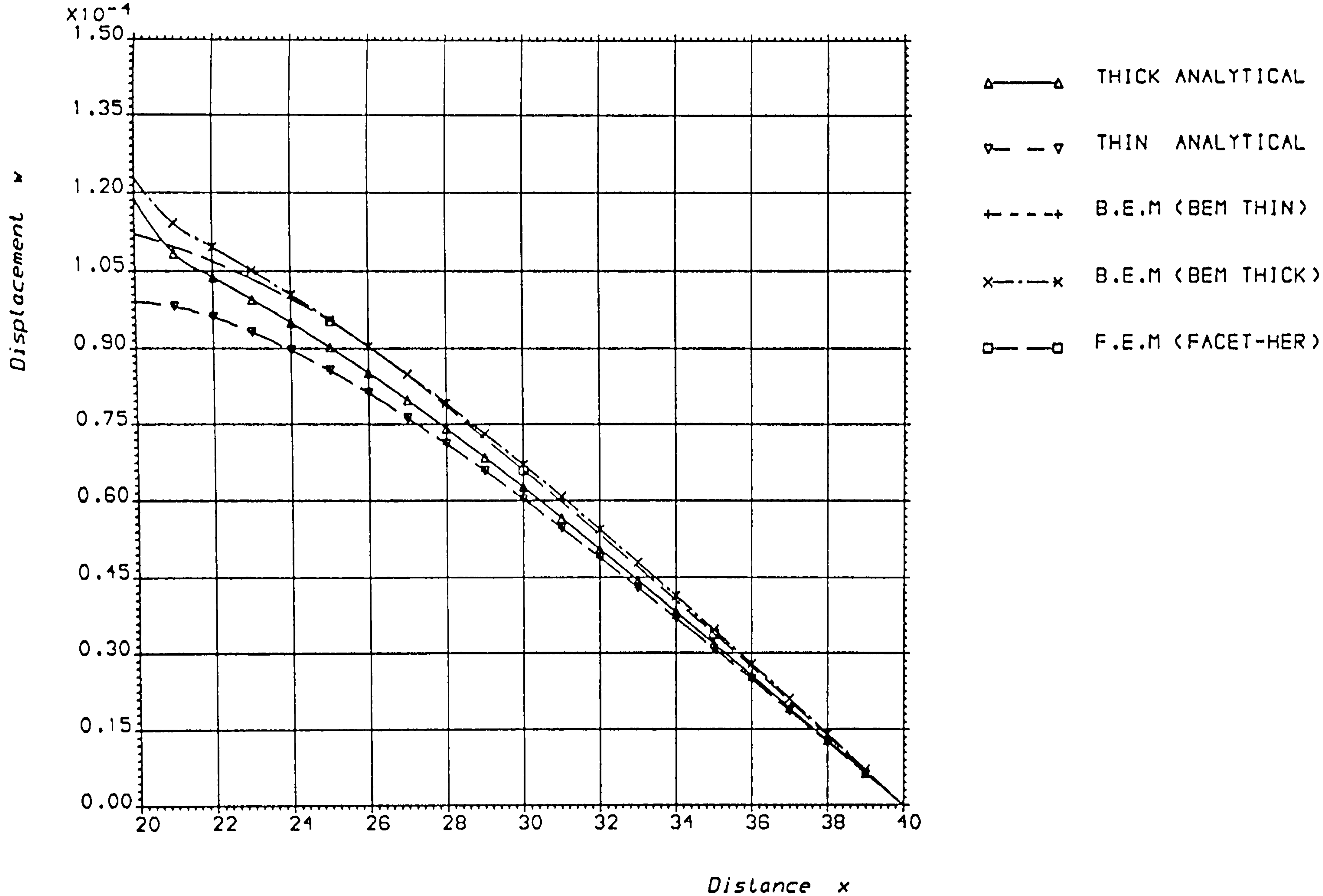


FIG (H.19) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(CONC. LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=4)

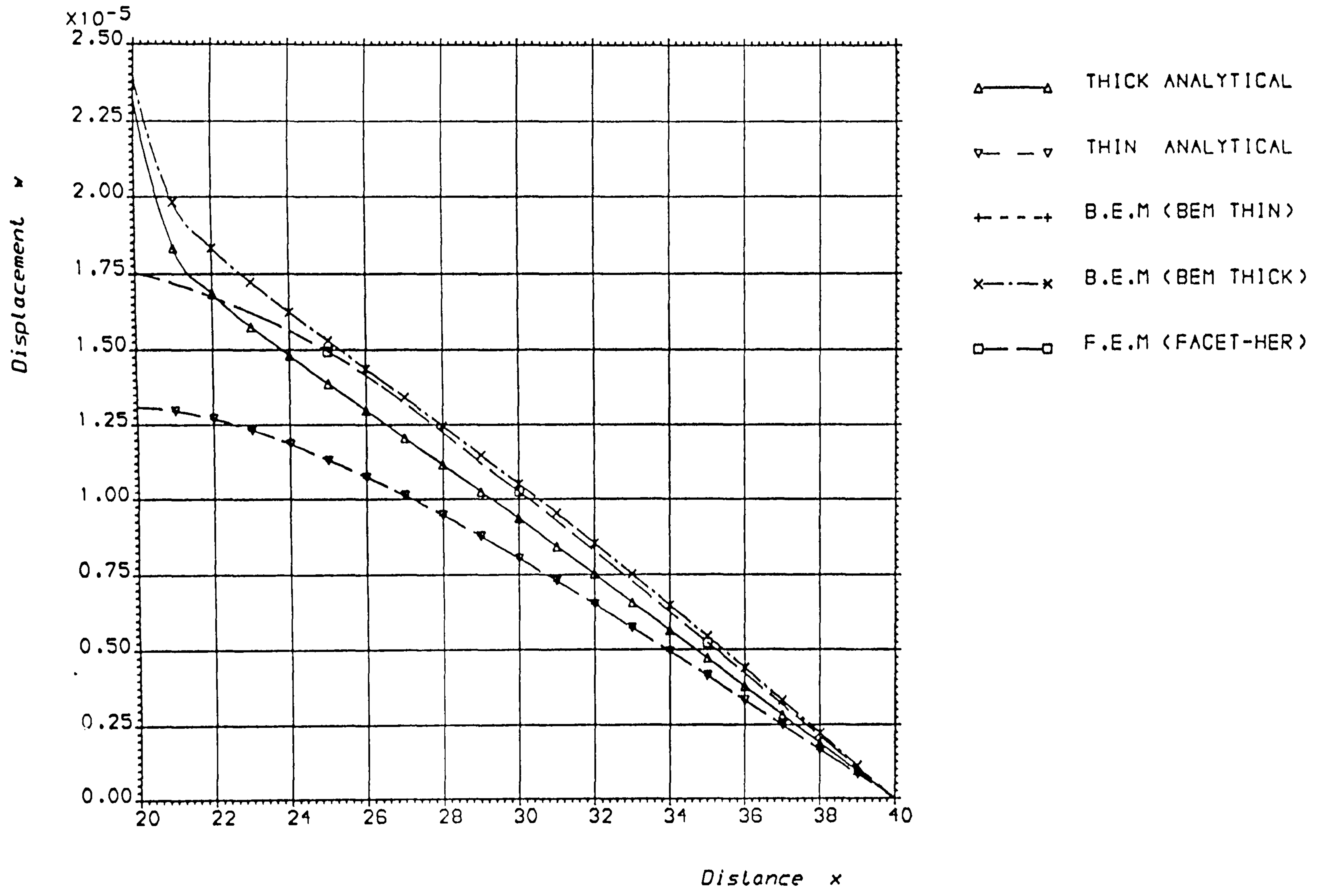


FIG (H.20) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(CONC. LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=8)

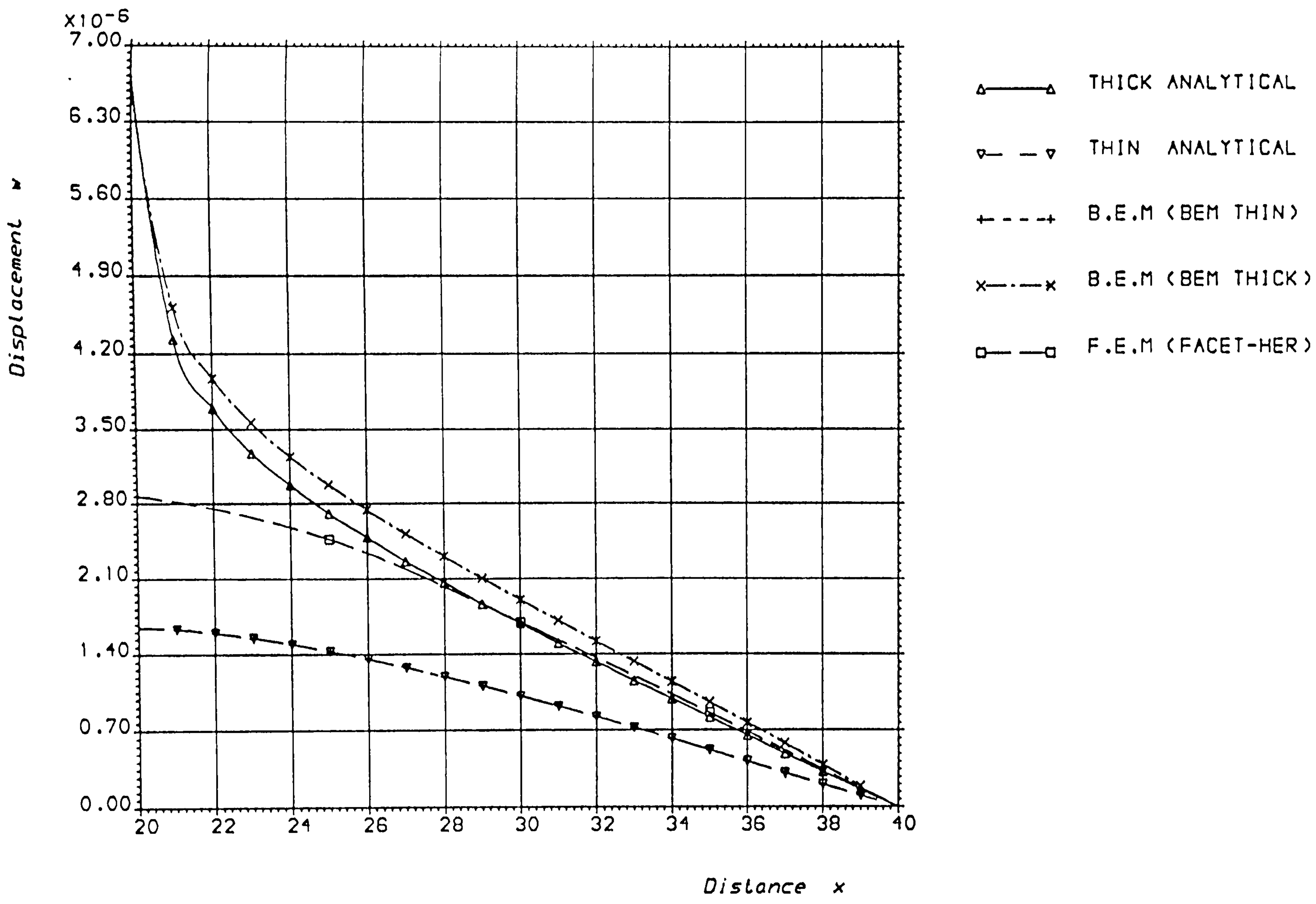


FIG (H.21) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE (CONC. LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=16)

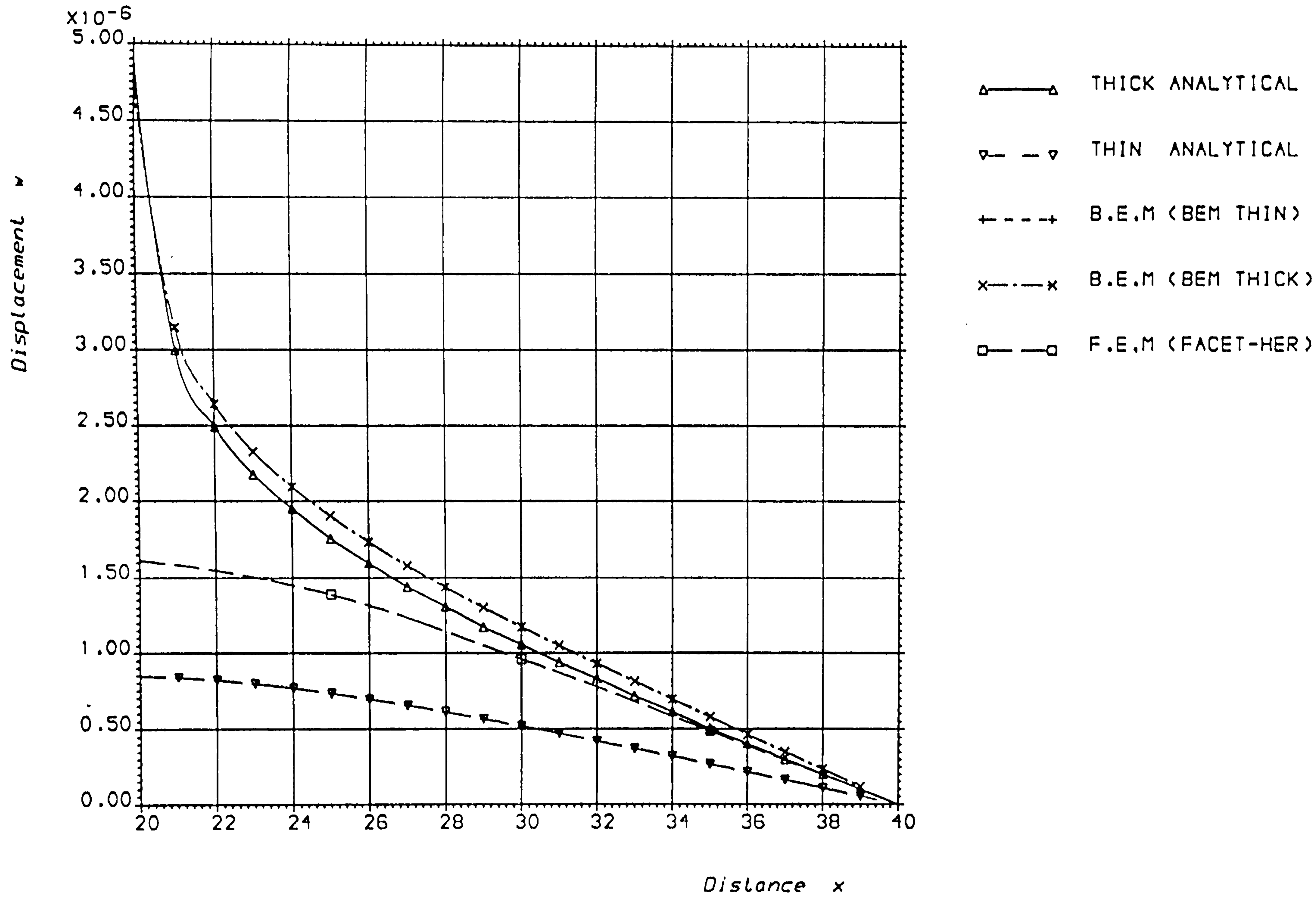


FIG (H.22) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE (CONC. LOADING CASE WITH FOUNDATION K=2000, THICKNESS h=20)

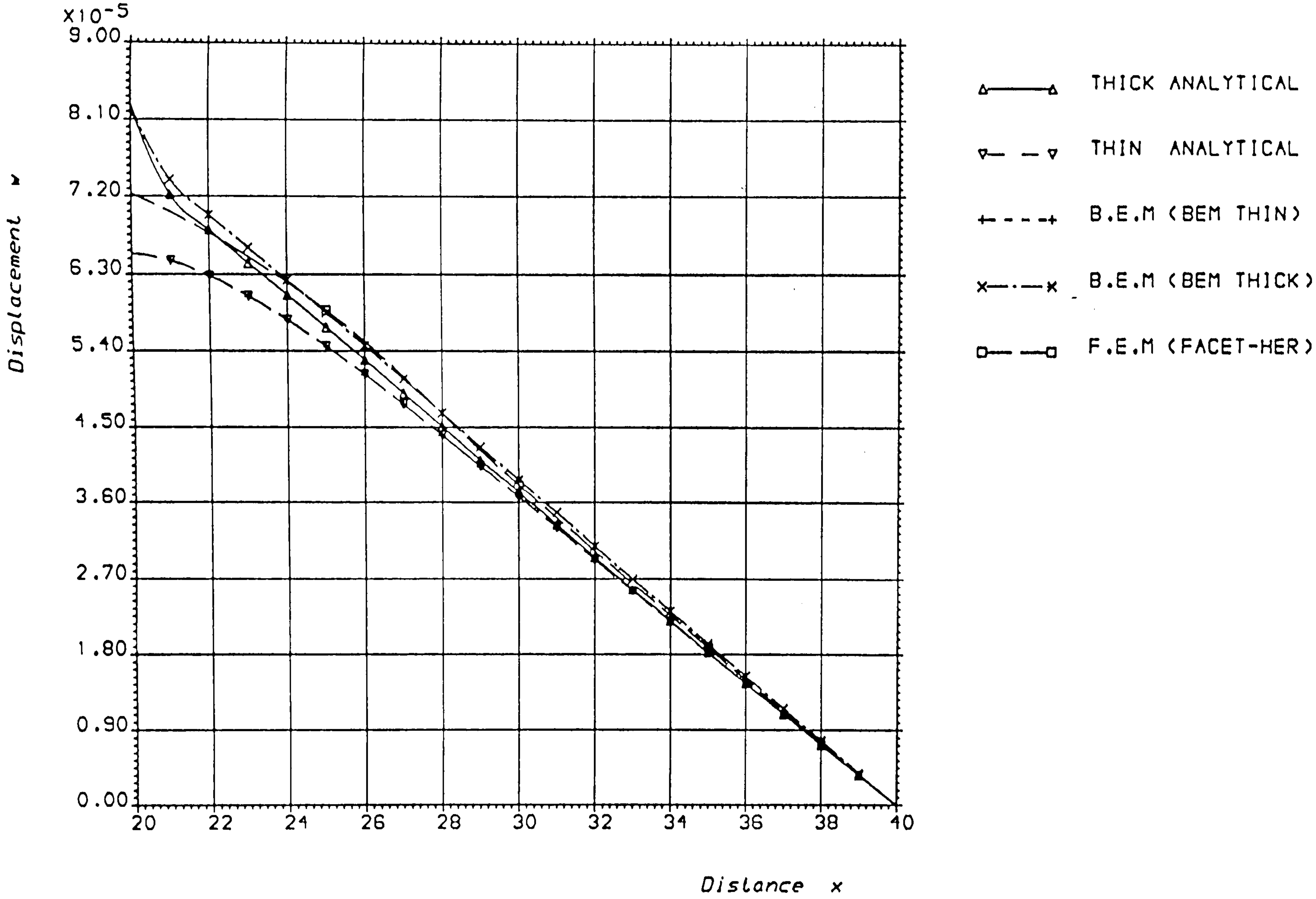


FIG (H.23) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(CONC. LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=4)

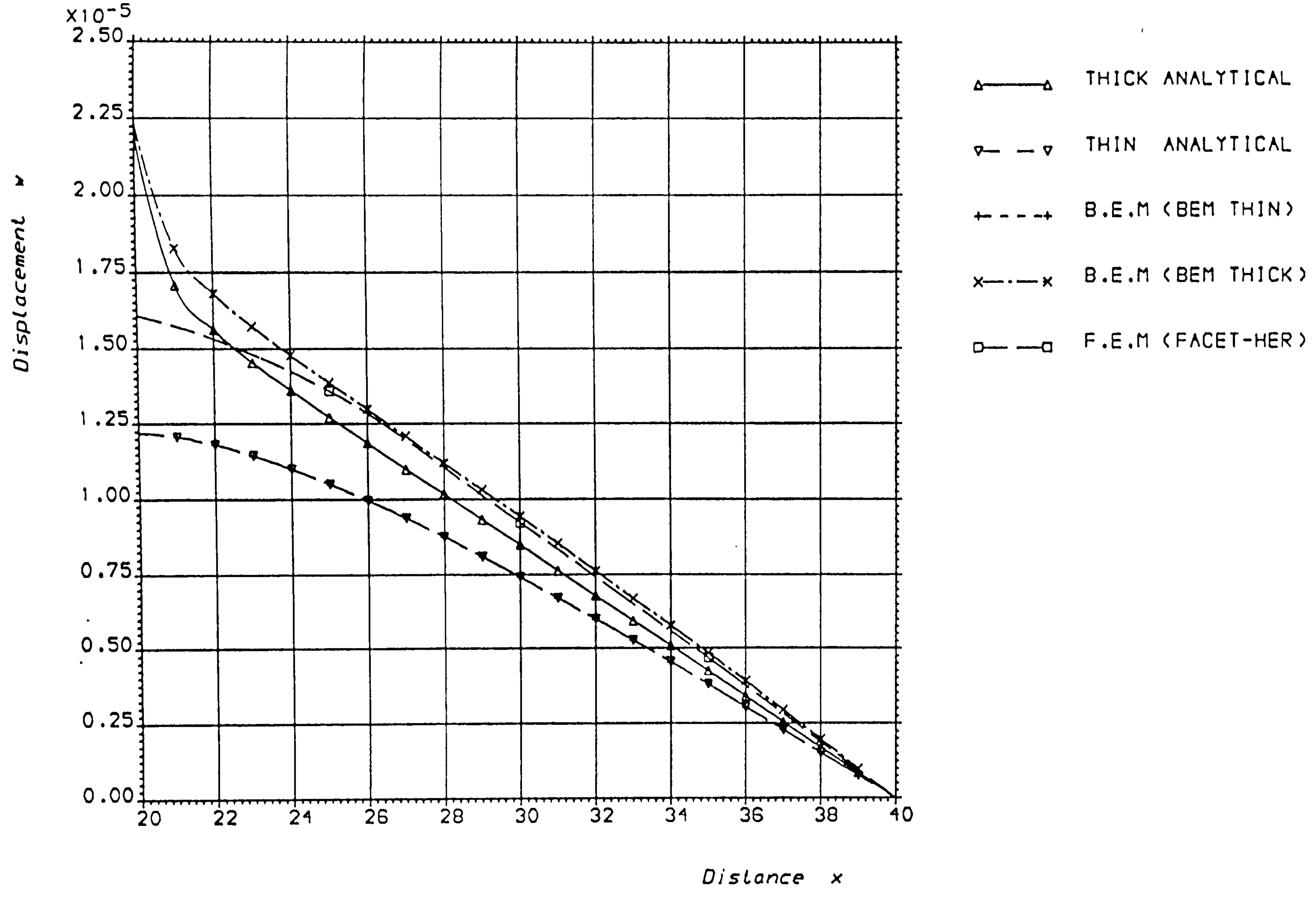


FIG (H.24) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(CONC. LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=8)

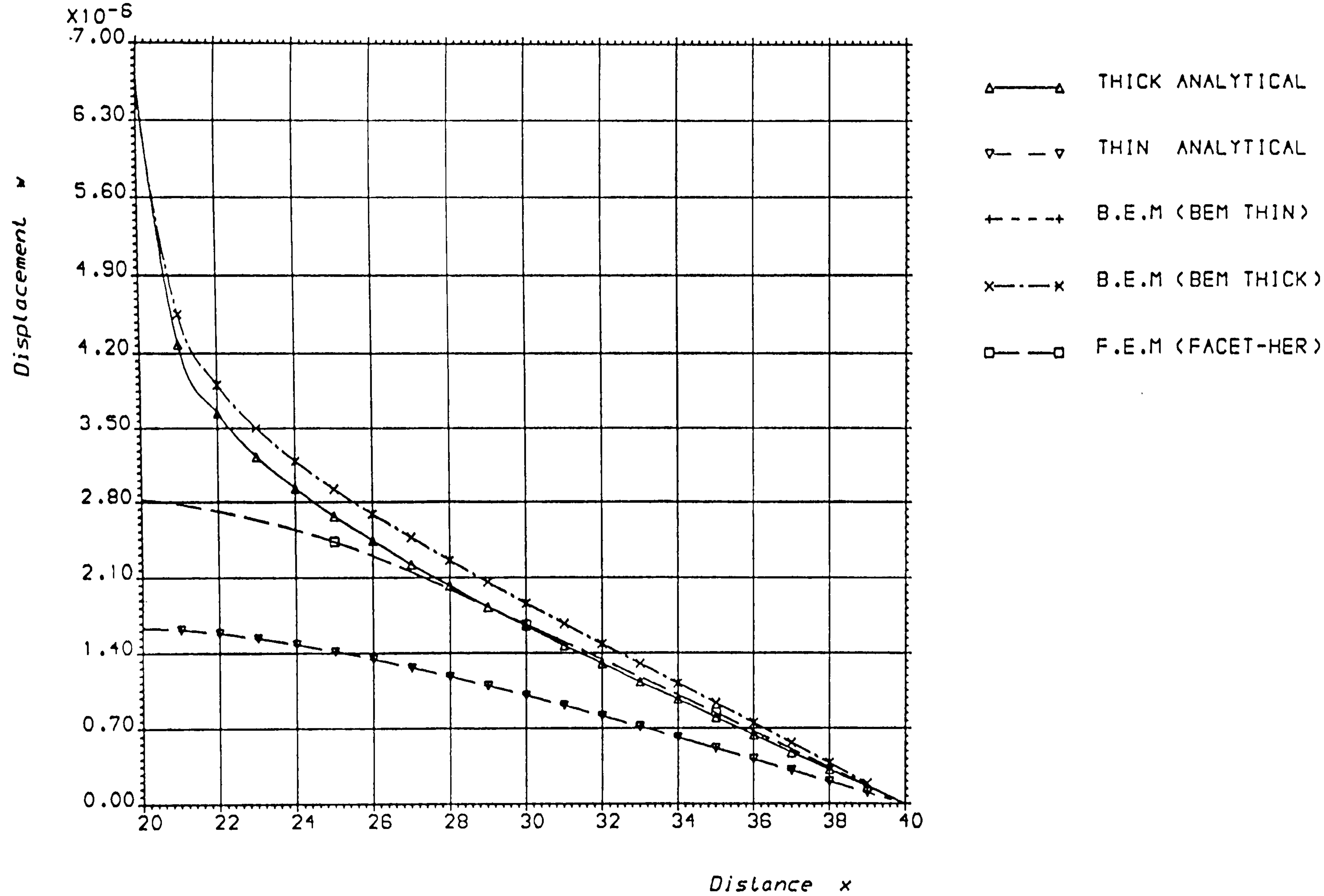


FIG (H.25) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(CONC. LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=16)

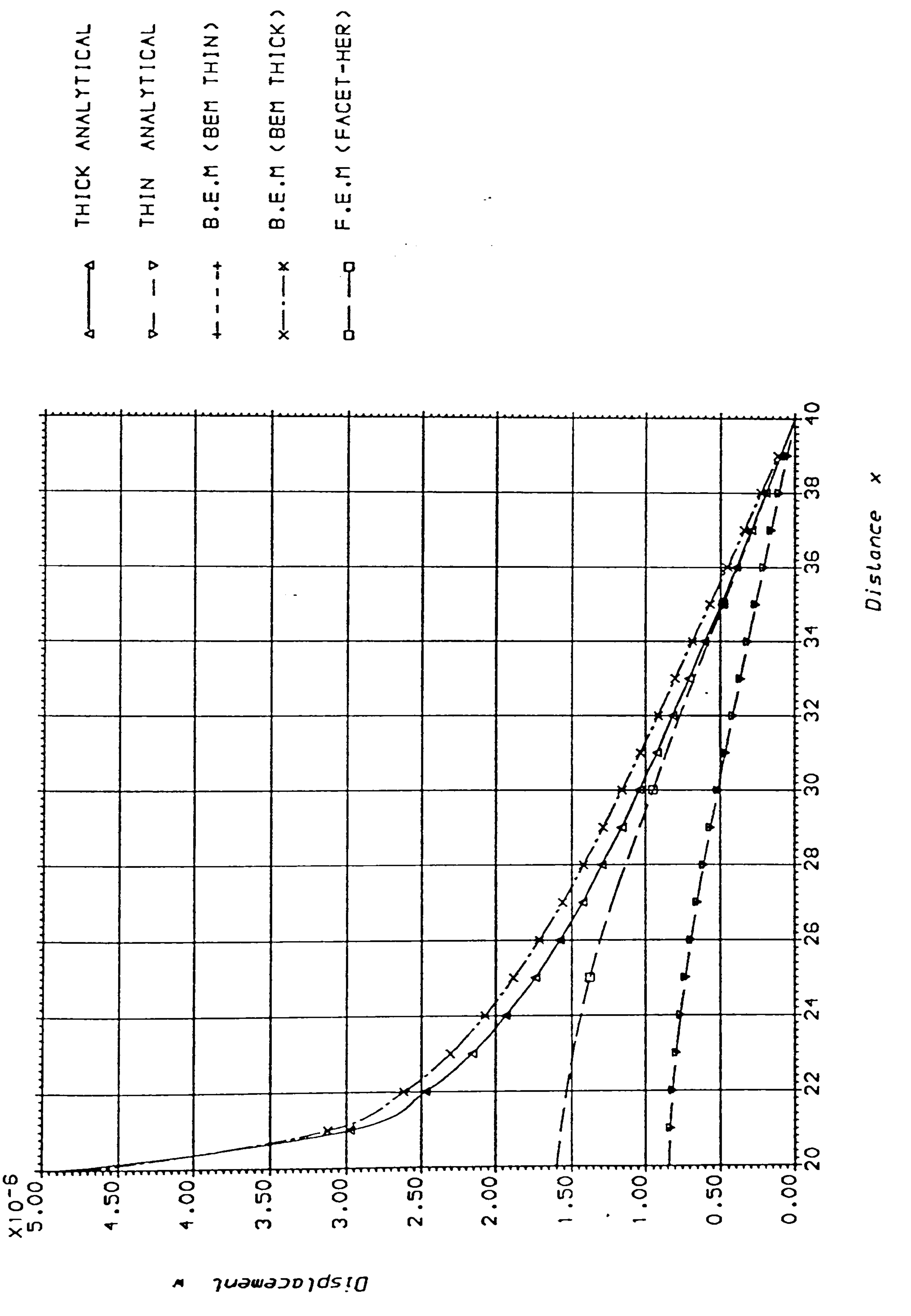


FIG (H.26) DISPLACEMENT DISTRIBUTION FOR SIMPLY-SUP RECTANGULR PLATE
(CONC. LOADING CASE WITH FOUNDATION K=20000, THICKNESS h=20)

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